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NOTES ON π-QUASI-NORMAL SUBGROUPS IN FINITE GROUPS

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ABSTRACT. Let G be a finite group and let π be a set of primes. A subgroup H of G is called π-quasi-normal in G if H permutes with every Sylow p-subgroup of G for every p in π. In this paper, we investigate how π-quasi-normality conditions on some subgroups of G affect the structure of G.

All groups considered are finite. The purpose of this paper is to investigate the influence of π-quasi-normality conditions on some subgroups of a finite group.

π denotes a set of primes. Let G be a group, and let H be a subgroup of G. H is called π-quasi-normal in G if H permutes with every Sylow p-subgroup of G for every p in π; H is called S-quasi-normal in G if H permutes with every Sylow subgroup of G; H is called quasi-normal in G if H permutes with every subgroup of G.

Lemma 1. *Let A, B, and C be subgroups of the group G. If A and B permute with C, then the subgroup $\langle A, B\rangle$ permutes with C* [1, *Hilfssatz* 1, *p.* 207].

Lemma 2. *Assume $A \leq M \leq G$ and $N \triangleleft G$. If A is π-quasi-normal in G, then A is π-quasi-normal in M and AN/N is π-quasi-normal in G/N* [1, *Hilfssatz* 3, *p.* 207].

Lemma 3. *Let A and B be subgroups in the group G. If A is π-quasi-normal in G and $AB = BA$, then $A \cap B$ is π-quasi-normal in B* [1, *Hilfssatz* 4, *p.* 207].

Lemma 4. *Let A be a subgroup in the group G. If A is S-quasi-normal in G, then $A \triangleleft\triangleleft G$* [1, *Satz* 1, *p.* 209].

Lemma 5. *Assume $A \leq G$ and $P \in \mathrm{Syl}_p(G)$ for every prime in p in π. If A is π-quasi-normal in G, then $A \cap P \in \mathrm{Syl}_p(A)$.*

Proof. Since AP is a subgroup of G, by the Sylow theorems, it follows that $A \cap P \in \mathrm{Syl}_p(A)$.

Lemma 6. *Let A be a maximal π-quasi-normal subgroup in G. Then one of the following statements is true:*

(a) *A is a maximal normal subgroup in G.*

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(b) $N := \langle \mathrm{Syl}_p(G) | p \in \pi \rangle \leq A$.
(c) *There exists a prime* p *in* π *such that* $AP = G$ *for every Sylow* p-*supgroup* P *of* G *and* $S := \langle \mathrm{Syl}_q(G) | q \in \pi \backslash \{p\} \rangle \leq A$.

Proof. If $A \triangleleft G$, then A is a maximal normal subgroup of G since A is maximal π-quasi-normal in G. We assume $A \ntriangleleft G$ and $N \nleq A$. AN is π-quasi-normal in G and $A < AN$, and so $AN = G$. Since $A \ntriangleleft G$, there exists a p-element $x \notin N_G(A)$, where p is a prime in π. Let $P \in \mathrm{Syl}_p(G)$ such that $x \in P$. We have $\langle A, A^x \rangle \leq AP = PA \leq G$. Obviously, A^x is also π-quasi-normal in G. Therefore, by Lemma 1, $\langle A, A^x \rangle$ is π-quasi-normal in G and $A < \langle A, A^x \rangle$. From this it follows that $AP = PA = G$. Thus, since A is π-quasi-normal in G, we have $AP = PA = G$ for every Sylow p-subgroup P of G. Let Q be any Sylow q-subgroup of G for every prime q in $\pi \backslash \{p\}$. Since $AP = G$, A contains a Sylow q-subgroup of G. Therefore, by Lemma 5, we have $Q \leq A$. This implies that $S \leq A$.

Corollary 1. *Let* A *be a maximal* p'-*quasi-normal subgroup in the group* G, *where* p *is a prime divisor of* $|G|$. *Then one of the following statements is true*:

(a) A *is maximal normal in* G.
(b) *There exists a prime* q *in* p' *such that* $AQ = QA = G$ *for every* $Q \in \mathrm{Syl}_q(G)$.

Proof. Suppose that $N := \langle \mathrm{Syl}_r(G) | r \in p' \rangle \leq A$; then $G = NP$ for every $p \in \mathrm{Syl}_p(G)$, and so $G = AP$ for every $P \in \mathrm{Syl}_p(G)$. It follows that A is S-quasi-normal in G. Thus, by Lemma 4, A is maximal normal in G. Using Lemma 6, the proof is complete.

From Lemma 6 we get immediately [2, Proposition 1, p. 211]

Corollary 2. *Suppose that* A *is a* π-*quasi-normal subgroup in the group* G; *then* $O_\pi(A) \leq O_\pi(G)$.

Proof. Induct on $|G|$. We can assume that $O_\pi(G) = 1$ and A is maximal π-quasi-normal in G. If $N := \langle \mathrm{Syl}_p(G) | p \in \pi \rangle \leq A$ then $O_\pi(A) \leq O_\pi(N) \leq O_\pi(G) = 1$. If $S := \langle \mathrm{Syl}_q(G) | q \in \pi \backslash \{p\} \rangle \leq A$ then $O^P(O_\pi(A)) \leq O_\pi(S) \leq O_\pi(G) = 1$, and so $O_\pi(A)$ is a p-group. Therefore, by Lemma 5, we have $O_\pi(A) \leq A \cap P \leq P$ for every $P \in \mathrm{Syl}_p(G)$, and so $O_\pi(A) \leq \bigcap_{g \in G} P^g = O_p(G) \leq O_\pi(G) = 1$. Hence, using Lemma 6, the proof is complete.

Theorem 1. *Let* A *and* B *be* π-*quasi-normal subgroups of* G *and* $G = AB$. *If* A *and* B *are* π-*solvable, then* G *is also* π-*solvable.*

Proof. Induct on $|G|$. Suppose that M is a π-quasi-normal subgroup of G and $A < M < G$; then $G = MB = BM$ and $M = M \cap G = M \cap AB = A(M \cap B)$. By Lemma 2 and Lemma 3, A and $M \cap B$ are π-quasi-normal in M. Thus M is π-solvable by induction. (Note: If M is a π'-group then M is π-solvable.) Therefore we can assume that A is maximal π-quasi-normal in G. By the same argument, we can assume that B is maximal π-quasi-normal in G.

We can assume: there is no π-solvable normal subgroup in G. Thus, by Lemma 6, we obtain that $\pi \cap \pi(G) = \{p\}$ and $G = AP = BP$ for every Sylow p-subgroup P of G. Then we have $O_\pi(G) = O_p(G)$, and so $O_\pi(G) = 1$. By Corollary 2, $O_\pi(A) \leq O_\pi(G) = 1$, so $O_{\pi'}(A) > 1$.

Since $G = AB$ and $AP = BP = G$, $|G : A \cap B|$ is a power of p. $A \cap B$ being π-solvable, there exists a Hall π'-subgroup H in $A \cap B$. Clearly H is a Hall π'-subgroup in A, B, and G as well. As $O_{\pi'}(A) \leq H$ we get

$$1 < O_{\pi'}(A)^G = O_{\pi'}(A)^{AB} = O_{\pi'}(A)^B \leq H^B \leq B < G.$$

Thus $O_{\pi'}(A)^G$ is a π-solvable normal subgroup in G—a contradiction.

For a p-solvable group X we denote by $\bar{r}_p(X)$ the arithmetical p-rank of X. $\bar{r}_p(X)$ is the least common multiple of the dimensions of the chief p-factors in X. (See [3, 8.2 Definition, p. 712].)

Let n be a positive integer, and let p be a prime. Let $\mathscr{F} = \{p\text{-solvable group } X | (|X|, n) = 1,\ \bar{r}_p(X) | n\}$. By [3, 8.3 Hilfssatz, p. 712], $\mathscr{F}$ is the formation defined locally by a system of formations $\{\mathscr{F}(q)\}$, where $\mathscr{F}(q)$ is defined as

(1) If $q|n$ then $\mathscr{F}(q) = \varnothing$;
(2) If $p \nmid n$ then $\mathscr{F}(p) = \{A | A \text{ is Abelian and } \exp(A) | p^n - 1\}$;
(3) If $q \neq p$ and $q \nmid n$ then $\mathscr{F}(q) =$ class of all groups.

By [3, 7.5 Hauptsatz, p. 697], $\mathscr{F}$ is a saturated formation.

Assume $X \in \mathscr{F}$. Let A be a subgroup of X. By [3, 7.4 Hilfssatz, p. 697], $X/F_p(X) \in \mathscr{F}(p)$. Since every group in $\mathscr{F}(p)$ is Abelian, we get $A/A \cap F_p(X) \in \mathscr{F}(p)$, and so $A/F_p(A) \in \mathscr{F}(p)$. Therefore, from [3, 7.4 Hilfssatz, p. 697] and the definition of $\{\mathscr{F}(q)\}$, it follows that $A \in \mathscr{F}$. This shows that $\mathscr{F}$ is closed with respect to taking subgroups.

Theorem 2. *Assume $G = AB$ with p'-quasi-normal subgroups A and B and $A, B \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if G' is p-nilpotent.*

Proof. (1) Assume $G \in \mathscr{F}$. By [3, 7.4 Hilfssatz, p. 697], $G/F_p(G) \in \mathscr{F}(p)$. Thus $G/F_p(G)$ is Abelian, and so $G' \leq F_p(G)$. As is well known, $F_p(G)$ is p-nilpotent, hence G' is also p-nilpotent.

(2) Let G' be p-nilpotent. We proceed by induction on $|G|$ to prove $G \in \mathscr{F}$. Since $\mathscr{F}$ is closed with respect to taking subgroups, by using the argument in Theorem 1, we can assume that A and B are maximal p'-quasi-normal in G.

By induction, we can assume $O_{p'}(G) = 1$. Then G' is a p-group, hence $G' \leq P$ for $P \in \mathrm{Syl}_p(G)$ and $P \triangleleft G$. From this it follows that G is solvable. Since A and B are maximal p'-quasi-normal in G and $G' \leq P$, A and B are maximal normal in G by Corollary 1. Therefore $r = |G : A|$ and $s = |G : B|$ are primes.

We assume $G \notin \mathscr{F}$. By induction, all proper quotient groups of G are $\mathscr{F}$-groups. Therefore the $\mathscr{F}$-residual $G_{\mathscr{F}}$ of G is a unique minimal normal subgroup of G, and it is an elementary Abelian p-group. By [3, 7.10 Satz, p. 700], we have $G = G_{\mathscr{F}} M$, where M is a covering (deckende) $\mathscr{F}$-subgroup of G. Obviously $G_{\mathscr{F}} \cap M = 1$. Since $P \triangleleft G$, $G_{\mathscr{F}} \leq Z(P)$. From this it follows that M is a Hall p'-subgroup of G and $G_{\mathscr{F}} = G' = P$. Hence, M is Abelian. In addition, $G_{\mathscr{F}}$ is a faithful and irreducible M-module, hence M is cyclic [4, Theorem 2.2, p. 64].

As $F_p(G) = P$, we have $F_p(G) = P = G_{\mathscr{F}} = G' \leq A \cap B$. Thus $r \neq s$ and $F_p(G) = F_p(A) = F_p(B)$. Since $A, B \in \mathscr{F}$, we have $A/F_p(A) \in \mathscr{F}(p)$ and $B/F_p(B) \in \mathscr{F}(p)$. It is well known that the product of two normal Abelian subgroups of coprime index is Abelian. Therefore we have $(A/F_p(A)) \cdot (B/F_p(B)) \in \mathscr{F}(p)$, yielding $G/F_p(G) \in \mathscr{F}(p)$. Thus, from [3, 7.4 Hilfssatz, p. 697] and the definition of $\{\mathscr{F}(q)\}$, it follows that $G \in \mathscr{F}$.

Corollary 3. *Assume $G = AB$ with p'-quasi-normal subgroups A and B. If A and B are p-supersolvable and G' is p-nilpotent, then G is p-supersolvable.*

Corollary 4. *Assume $G = AB$ with S-quasi-normal subgroups A and B. If A and B are supersolvable and G' is nilpotent, then G is supersolvable.*

In particular, if A and B are two quasi-normal supersolvable subgroups and $(AB)'$ is nilpotent, then AB is a quasi-normal supersolvable subgroup.

The next statement improves [2, Theorem 3, p. 214].

Corollary 5. *Assume $G = AB$ with p'-quasi-normal p-solvable subgroups A and B and $(\bar{r}_p(A) \cdot \bar{r}_p(B), |G|) = 1$. Then G is p-solvable. Further, $\bar{r}_p(G) = [\bar{r}_p(A), \bar{r}_p(B)]$ if and only if G' is p-nilpotent.*

($[a, b]$ denotes the least common multiple of the positive integers a and b.)

Proof. (1) Assume $\bar{r}_p(G) = [\bar{r}_p(A), \bar{r}_p(B)]$. Let $n = \bar{r}_p(G)$. Then G' is p-nilpotent by Theorem 2. (Note: G is p-solvable by [5, Theorem 1, p. 229].)

(2) Let G' be p-nilpotent, and let $n = [\bar{r}_p(A), \bar{r}_p(B)]$. By Theorem 2, G is p-solvable and $\bar{r}_p(G) | [\bar{r}_p(A), \bar{r}_p(B)]$. On the other hand, taking $n = \bar{r}_p(G)$, we have $(n, |G|) = 1$ since $\bar{r}_p(G) | [\bar{r}_p(A), \bar{r}_p(B)]$ and $(\bar{r}_p(A) \cdot \bar{r}_p(B), |G|) = 1$. Thus we have $G \in \mathscr{F}$. Since $\mathscr{F}$ is closed with respect to taking subgroups, we get $A, B \in \mathscr{F}$. Thus $\bar{r}_p(A) | n = \bar{r}_p(G)$ and $\bar{r}_p(B) | n = \bar{r}_p(G)$, and so $[\bar{r}_p(A), \bar{r}_p(B)] | \bar{r}_p(G)$. It follows that $[\bar{r}_p(A), \bar{r}_p(B)] = \bar{r}_p(G)$.

Let G be a group. We say G has property T_π, if $A \triangleleft B \leq G$ then $A = B \cap H$ with H π-quasi-normal in G.

If G has T_π, it is easy to check that every normal subgroup and quotient of G has T_π as well.

Theorem 3. *Let G be a group, and let p be a prime divisor of $|G|$. If G has $T_{p'}$, then the following statements are true:*

(1) G is p-supersolvable and $l_p(G) = 1$. ($l_p(G)$ stands for p-length of G.)

(2) If $p \not\equiv 1 \pmod q$ for every prime divisor q of $|G|$, then G possesses a supersolvable normal p-complement.

(3) If $N_G(P)$ is p-nilpotent, where P is a Sylow p-subgroup of G, then G possesses a supersolvable normal p-complement.

Proof. (1) First, by induction on $|G|$, we prove that G is p-supersolvable. By [5, Lemma 4, p. 230], G is p-solvable. We can assume $O_{p'}(G) = 1$. In addition, by [3, 8.6 Satz, p. 713], we can assume $\Phi(G) = 1$. Hence we have $G = LM$, where L is a minimal normal elementary Abelian p-group and M is a maximal subgroup of G with $L \cap M = 1$. Let $R \in \mathrm{Syl}_p(M)$; then $P = LR \in \mathrm{Syl}_p(G)$. If A stands for a maximal subgroup of P containing R, then $A = R(A \cap L)$. By $T_{p'}$, $A = P \cap K$ with K p'-quasi-normal in G. Let $Q \in \mathrm{Syl}_q(G)$ for every $q \in p'$. We have $KQ = QK \leq G$, hence $L \cap K = L \cap KQ \triangleleft KQ = QK$. It follows that $\langle \mathrm{Syl}_q(G) | q \neq p \rangle \triangleleft N_G(L \cap K) = N_G(L \cap P \cap K) = N_G(L \cap A)$. Clearly, $G = \langle \mathrm{Syl}_q(G) | q \neq p \rangle \cdot P \leq N_G(L \cap A)$. So $L \cap A = 1$, whence $|L| = p$. On the other hand, by induction, G/L is p-supersolvable, hence G is p-supersolvable.

Now we prove $l_p(G) = 1$. We can assume $O_{p'}(G) = 1$. Then, since G is p-supersolvable, G' is a p-group by [3, 9.1 Satz, p. 716]. From this it follows that $P \triangleleft G$, where P is a Sylow p-subgroup of G. This implies that the p-length of G is 1.

(2) Let $Q \in \mathrm{Syl}_q(G)$ for each $q \in p'$, and let V be any chief p-factor of G. Since G is p-supersolvable, $|V| = p$. It follows that $Q \leq F_p(G)$ since $p \not\equiv 1 \pmod q$. As is well known, $F_p(G)$ is p-nilpotent, hence G is p-nilpotent.

We have $G = PH$, where P is a Sylow p-group of G and H is a normal p-complement of G. Since H has $T_{p'}$ and $p \notin \pi(H)$, H has T_π with $\pi = \pi(H)$. Therefore H is supersolvable by (1).

(3) We can assume $O_{p'}(G) = 1$. Thus, by (1), we have $P \triangleleft G$. Since $N_G(P)$ is p-nilpotent, $G = N_G(P)$ is p-nilpotent.

Corollary 6 [5, Corollary 4, p. 232]. *A π-group satisfying T_π is supersolvable.*

As an assertion similar to Theorem 3, we mention the following result.

Theorem 4. *Let G be a group, and let p be a prime divisor of $|G|$. Suppose that each maximal subgroup of a Sylow p-subgroup P of G is p'-quasi-normal in G. If $p \not\equiv 1 \pmod q$ for every prime divisor q of $|G|$, then G is p-nilpotent.*

Proof. By Sylow's theorem, each maximal subgroup of every Sylow p-subgroup of G is p'-quasi-normal in G. It easy to check that every quotient of G satisfies the hypothesis of the theorem.

Let $Q \in \mathrm{Syl}_q(G)$ for every $q \in p'$. If P is cyclic, then $N_G(P) = C_G(P)$, and so G is p-nilpotent by [4, Theorem 4.3, p. 252]. Hence we assume that P is not cyclic. Then P is the product of two maximal subgroups of it, and so P is p'-quasi-normal in G. Therefore we have $PQ = QP \leq G$. The subgroup PQ is solvable and PQ satisfies the hypothesis of the theorem by Lemma 2. Thus, by using the argument in Theorem 3, we conclude that the subgroup PQ is p-nilpotent. Then we have $P \leq N_G(Q)$. From this and [4, Theorem 4.5, p. 253], it follows that G is p-nilpotent.

Corollary 7. *Let G be a group and let p be the smallest prime divisor of $|G|$. If every maximal subgroup of a Sylow p-subgroup P of G is p'-quasi-normal in G, then G is p-nilpotent.*

Corollary 8. *Let G be a group, and let $p_1 < p_2 < \cdots < p_r$ be the distinct primes dividing $|G|$. If each maximal subgroup of Sylow p_1-subgroups of G is $\{p_{i+1}, \ldots, p_r\}$-quasi-normal in G, $i = 1, \ldots, r-1$, then G satisfies the Sylow tower property.*

Corollary 9. *Let G be a group. If each maximal subgroup of a Sylow p-subgroup of G is p'-quasi-normal in G for every prime divisor of p of $|G|$, then G is supersolvable.*

Proof. By Corollary 8, G is solvable. Thus, by using the argument in the proof (1) of Theorem 3, it follows that G is supersolvable.

Corollary 10 (Srinivasan [6]). *Let G be a group. If each maximal subgroup of Sylow subgroups of G is S-quasi-normal in G, then G is supersolvable.*

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