NOTES ON $\pi$-QUASI-NORMAL SUBGROUPS IN FINITE GROUPS

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(Communicated by Warren J. Wong)

Abstract. Let $G$ be a finite group and let $\pi$ be a set of primes. A subgroup $H$ of $G$ is called $\pi$-quasi-normal in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ for every $p$ in $\pi$. In this paper, we investigate how $\pi$-quasi-normality conditions on some subgroups of $G$ affect the structure of $G$.

All groups considered are finite. The purpose of this paper is to investigate the influence of $\pi$-quasi-normality conditions on some subgroups of a finite group.

$\pi$ denotes a set of primes. Let $G$ be a group, and let $\pi$ be a subgroup of $G$. $\pi$ is called $\pi$-quasi-normal in $G$ if $\pi$ permutes with every Sylow $p$-subgroup of $G$ for every $p$ in $\pi$; $\pi$ is called $S$-quasi-normal in $G$ if $\pi$ permutes with every Sylow subgroup of $G$; $\pi$ is called quasi-normal in $G$ if $\pi$ permutes with every subgroup of $G$.

Lemma 1. Let $A$, $B$, and $C$ be subgroups of the group $G$. If $A$ and $B$ permute with $C$, then the subgroup $(A, B)$ permutes with $C$ [1, Hilfssatz 1, p. 207].

Lemma 2. Assume $A \leq M \leq G$ and $N \triangleleft G$. If $A$ is $\pi$-quasi-normal in $G$, then $A$ is $\pi$-quasi-normal in $M$ and $AN/N$ is $\pi$-quasi-normal in $G/N$ [1, Hilfssatz 3, p. 207].

Lemma 3. Let $A$ and $B$ be subgroups in the group $G$. If $A$ is $\pi$-quasi-normal in $G$ and $AB = BA$, then $A \cap B$ is $\pi$-quasi-normal in $B$ [1, Hilfssatz 4, p. 207].

Lemma 4. Let $A$ be a subgroup in the group $G$. If $A$ is $S$-quasi-normal in $G$, then $A \triangleleft G$ [1, Satz 1, p. 209].

Lemma 5. Assume $A \leq G$ and $P \in \operatorname{Syl}_p(G)$ for every prime in $p$ in $\pi$. If $A$ is $\pi$-quasi-normal in $G$, then $A \cap P \in \operatorname{Syl}_p(A)$.

Proof. Since $AP$ is a subgroup of $G$, by the Sylow theorems, it follows that $A \cap P \in \operatorname{Syl}_p(A)$.

Lemma 6. Let $A$ be a maximal $\pi$-quasi-normal subgroup in $G$. Then one of the following statements is true:

(a) $A$ is a maximal normal subgroup in $G$.
(b) \( N := \langle \text{Syl}_p(G) \mid p \in \pi \rangle \leq A \).

(c) There exists a prime \( p \) in \( \pi \) such that \( AP = G \) for every Sylow \( p \)-subgroup \( P \) of \( G \) and \( S := \langle \text{Syl}_q(G) \mid q \in \pi \setminus \{p\} \rangle \leq A \).

**Proof.** If \( A \triangleleft G \), then \( A \) is a maximal normal subgroup of \( G \) since \( A \) is maximal \( \pi \)-quasi-normal in \( G \). We assume \( A \not\triangleleft G \) and \( N \not\triangleleft A \). \( AN \) is \( \pi \)-quasi-normal in \( G \) and \( A < AN \), and so \( AN = G \). Since \( A \not\triangleleft G \), there exists a \( p \)-element \( x \not\in N_G(A) \), where \( p \) is a prime in \( \pi \). Let \( P \in \text{Syl}_p(G) \) such that \( x \in P \). We have \( \langle A, A^x \rangle \leq AP = PA \leq G \). Obviously, \( A^x \) is also \( \pi \)-quasi-normal in \( G \). Therefore, by Lemma 1, \( \langle A, A^x \rangle \) is \( \pi \)-quasi-normal in \( G \) and \( A < \langle A, A^x \rangle \). From this it follows that \( AP = PA = G \). Thus, since \( A \) is \( \pi \)-quasi-normal in \( G \), we have \( AP = PA = G \) for every Sylow \( p \)-subgroup \( P \) of \( G \). Let \( Q \) be any Sylow \( q \)-subgroup of \( G \) for every prime \( q \) in \( \pi \setminus \{p\} \). Since \( AP = G \), \( A \) contains a Sylow \( q \)-subgroup of \( G \). Therefore, by Lemma 5, we have \( Q \leq A \). This implies that \( S \leq A \).

**Corollary 1.** Let \( A \) be a maximal \( p' \)-quasi-normal subgroup in the group \( G \), where \( p \) is a prime divisor of \( |G| \). Then one of the following statements is true:

(a) \( A \) is maximal normal in \( G \).

(b) There exists a prime \( q \) in \( p' \) such that \( AQ = QA = G \) for every \( Q \in \text{Syl}_q(G) \).

**Proof.** Suppose that \( N := \langle \text{Syl}_p(G) \mid r \in p' \rangle \leq A \); then \( G = NP \) for every \( p \in \text{Syl}_p(G) \), and so \( G = AP \) for every \( P \in \text{Syl}_p(G) \). It follows that \( A \) is \( S \)-quasi-normal in \( G \). Thus, by Lemma 4, \( A \) is maximal normal in \( G \). Using Lemma 6, the proof is complete.

From Lemma 6 we get immediately [2, Proposition 1, p. 211]

**Corollary 2.** Suppose that \( A \) is a \( \pi \)-quasi-normal subgroup in the group \( G \); then \( O_{\pi}(A) \leq O_{\pi}(G) \).

**Proof.** Induct on \( |G| \). We can assume that \( O_{\pi}(G) = 1 \) and \( A \) is maximal \( \pi \)-quasi-normal in \( G \). If \( N := \langle \text{Syl}_p(G) \mid p \in \pi \rangle \leq A \) then \( O_{\pi}(A) \leq O_{\pi}(N) \leq O_{\pi}(G) = 1 \). If \( S := \langle \text{Syl}_q(G) \mid q \in \pi \setminus \{p\} \rangle \leq A \) then \( O_{\pi}(O_{\pi}(A)) \leq O_{\pi}(S) \leq O_{\pi}(G) = 1 \), and so \( O_{\pi}(A) \) is a \( p \)-group. Therefore, by Lemma 5, we have \( O_{\pi}(A) \leq A \cap P \leq P \) for every \( P \in \text{Syl}_p(G) \), and so \( O_{\pi}(A) \leq \bigcap_{g \in G} p^g = O_p(G) \leq O_{\pi}(G) = 1 \). Hence, using Lemma 6, the proof is complete.

**Theorem 1.** Let \( A \) and \( B \) be \( \pi \)-quasi-normal subgroups of \( G \) and \( G = AB \). If \( A \) and \( B \) are \( \pi \)-solvable, then \( G \) is also \( \pi \)-solvable.

**Proof.** Induct on \( |G| \). Suppose that \( M \) is a \( \pi \)-quasi-normal subgroup of \( G \) and \( A < M < G \); then \( G = MB = BM \) and \( M = M \cap G = M \cap AB = A(M \cap B) \). By Lemma 2 and Lemma 3, \( A \) and \( M \cap B \) are \( \pi \)-quasi-normal in \( M \). Thus \( M \) is \( \pi \)-solvable by induction. (Note: If \( M \) is a \( p' \)-group then \( M \) is \( \pi \)-solvable.) Therefore we can assume that \( A \) is maximal \( \pi \)-quasi-normal in \( G \). By the same argument, we can assume that \( B \) is maximal \( \pi \)-quasi-normal in \( G \).

We can assume: there is no \( \pi \)-solvable normal subgroup in \( G \). Thus, by Lemma 6, we obtain that \( \pi \cap \pi(G) = \{p\} \) and \( G = AP = BP \) for every Sylow \( p \)-subgroup \( P \) of \( G \). Then we have \( O_{\pi}(G) = O_p(G) \), and so \( O_{\pi}(G) = 1 \). By Corollary 2, \( O_{\pi}(A) \leq O_{\pi}(G) = 1 \), so \( O_{\pi}(A) > 1 \).
Since $G = AB$ and $A \cap B \triangleleft G$, the chief factor $A \cap B$ is a power of $p$. Let $A \cap B$ be a Hall $\pi$-subgroup in $G$ and $A \cap B \triangleleft G$ as well. As $O_{\pi'}(A) \leq H$ we get
\[ 1 < O_{\pi'}(A)^G = O_{\pi'}(A)^A B = O_{\pi'}(A)^B \leq H^B \leq B < G. \]
Thus $O_{\pi'}(A)^G$ is a $\pi$-solvable normal subgroup in $G$ — a contradiction.

For a $p$-solvable group $X$ we denote by $\bar{\rho}(X)$ the arithmetical $p$-rank of $X$. The $\bar{\rho}(X)$ is the least common multiple of the dimensions of the chief $p$-factors in $X$. (See [3, 8.2 Definition, p. 712].)

Let $n$ be a positive integer, and let $p$ be a prime. Let $\mathcal{F} = \{p$-solvable group $X([X], n) = 1, \bar{\rho}(X)|n\}$. By [3, 8.3 Hilfssatz, p. 712], $\mathcal{F}$ is the formation defined locally by a system of formations $\{\mathcal{F}(q)\}$, where $\mathcal{F}(q)$ is defined as
\begin{enumerate}
  \item If $q \nmid n$ then $\mathcal{F}(q) = \emptyset$;
  \item If $p \nmid n$ then $\mathcal{F}(p) = \{A | A$ is Abelian and $\exp(A)|p^n - 1\}$;
  \item If $q \parallel p$ and $q \nmid n$ then $\mathcal{F}(q) =$ class of all groups.
\end{enumerate}

By [3, 7.5 Hauptsatz, p. 697], $\mathcal{F}$ is a saturated formation.

Assume $X \in \mathcal{F}$. Let $A$ be a subgroup of $X$. By [3, 7.4 Hilfssatz, p. 697], $X/F_p(X) \in \mathcal{F}(p)$. Since every group in $\mathcal{F}(p)$ is Abelian, we get $A/A \cap F_p(X) \in \mathcal{F}(p)$, and so $A/F_p(A) \in \mathcal{F}(p)$. Therefore, from [3, 7.4 Hilfssatz, p. 697] and the definition of $\{\mathcal{F}(q)\}$, it follows that $A \in \mathcal{F}$. This shows that $\mathcal{F}$ is closed with respect to taking subgroups.

Theorem 2. Assume $G = AB$ with $p'$-quasi-normal subgroups $A$ and $B$ and $A, B \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if $G'$ is $p$-nilpotent.

Proof. (1) Assume $G \in \mathcal{F}$. By [3, 7.4 Hilfssatz, p. 697], $G/F_p(G) \in \mathcal{F}(p)$. Thus $G/F_p(G)$ is Abelian, and so $G' \leq F_p(G)$. As is well known, $F_p(G)$ is $p$-nilpotent, hence $G'$ is also $p$-nilpotent.

(2) Let $G'$ be $p$-nilpotent. We proceed by induction on $|G|$ to prove $G \in \mathcal{F}$. Since $\mathcal{F}$ is closed with respect to taking subgroups, by using the argument in Theorem 1, we can assume that $A$ and $B$ are maximal $p'$-quasi-normal in $G$.

By induction, we can assume $O_p(G) = 1$. Then $G'$ is a $p$-group, hence $G' \leq P$ for $P \in \text{Syl}_p(G)$ and $P \triangleleft G$. From this it follows that $G$ is solvable. Since $A$ and $B$ are maximal $p'$-quasi-normal in $G$ and $G' \leq P$, $A$ and $B$ are maximal normal in $G$ by Corollary 1. Therefore $r = |G : A|$ and $s = |G : B|$ are primes.

We assume $G \notin \mathcal{F}$. By induction, all proper quotient groups of $G$ are $\mathcal{F}$-groups. Therefore the $\mathcal{F}$-residual $G_{\mathcal{F}}$ of $G$ is a unique minimal normal subgroup of $G$, and it is an elementary Abelian $p$-group. By [3, 7.10 Satz, p. 700], we have $G = G_{\mathcal{F}}M$, where $M$ is a covering (deckende) $\mathcal{F}$-subgroup of $G$. Obviously $G_{\mathcal{F}} \cap M = 1$. Since $P \triangleleft G$, $G_{\mathcal{F}} \leq Z(P)$. From this it follows that $M$ is a Hall $p'$-subgroup of $G$ and $G_{\mathcal{F}} = G' = P$. Hence, $M$ is Abelian. In addition, $G_{\mathcal{F}}$ is a faithful and irreducible $M$-module, hence $M$ is cyclic [4, Theorem 2.2, p. 64].

As $F_p(G) = P$, we have $F_p(G) = P = G_{\mathcal{F}} = G' \leq A \cap B$. Thus $r \neq s$ and $F_p(G) = F_p(A) = F_p(B)$. Since $A, B \in \mathcal{F}$, we have $A/F_p(A) \in \mathcal{F}(p)$ and $B/F_p(B) \in \mathcal{F}(p)$. It is well known that the product of two normal Abelian subgroups of coprime index is Abelian. Therefore we have $(A/F_p(A))(B/F_p(B)) \in \mathcal{F}(p)$, yielding $G/F_p(G) \in \mathcal{F}(p)$. Thus, from [3, 7.4 Hilfssatz, p. 697] and the definition of $\{\mathcal{F}(q)\}$, it follows that $G \in \mathcal{F}$. 

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Corollary 3. Assume $G = AB$ with $p'$-quasi-normal subgroups $A$ and $B$. If $A$ and $B$ are $p$-supersolvable and $G'$ is $p$-nilpotent, then $G$ is $p$-supersolvable.

Corollary 4. Assume $G = AB$ with $S$-quasi-normal subgroups $A$ and $B$. If $A$ and $B$ are supersolvable and $G'$ is nilpotent, then $G$ is supersolvable.

In particular, if $A$ and $B$ are two quasi-normal supersolvable subgroups and $(AB)'$ is nilpotent, then $AB$ is a quasi-normal supersolvable subgroup.

The next statement improves [2, Theorem 3, p. 214].

Corollary 5. Assume $G = AB$ with $p'$-quasi-normal $p$-solvable subgroups $A$ and $B$ and $(T_p(A) \cdot T_p(B), |G|) = 1$. Then $G$ is $p$-solvable. Further, $T_p(G) = [T_p(A), T_p(B)]$ if and only if $G'$ is $p$-nilpotent.

Proof. (1) Assume $T_p(G) = [T_p(A), T_p(B)]$. Let $n = T_p(G)$. Then $G'$ is $p$-nilpotent by Theorem 2. (Note: $G$ is $p$-solvable by [5, Theorem 1, p. 229].)

(2) Let $G'$ be $p$-nilpotent, and let $n = [T_p(A), T_p(B)]$. By Theorem 2, $G$ is $p$-solvable and $T_p(G)[T_p(A), T_p(B)]$. On the other hand, taking $n = T_p(G)$, we have $(n, |G|) = 1$ since $T_p(G)[T_p(A), T_p(B)]$ and $(T_p(A) \cdot T_p(B), |G|) = 1$. Thus we have $G \in T$. Since $T$ is closed with respect to taking subgroups, we get $A, B \in T$. Thus $T_p(A)n = T_p(G)$ and $T_p(B)n = T_p(G)$, and so $[T_p(A), T_p(B)]T_p(G)$. It follows that $[T_p(A), T_p(B)] = T_p(G)$.

Let $G$ be a group. We say $G$ has property $T_\pi$, if $A < B < G$ then $A = T_p \cap H$ with $H$ $\pi$-quasi-normal in $G$.

If $G$ has $T_\pi$, it is easy to check that every normal subgroup and quotient of $G$ has $T_\pi$ as well.

Theorem 3. Let $G$ be a group, and let $p$ be a prime divisor of $|G|$. If $G$ has $T_p'$, then the following statements are true:

(1) $G$ is $p$-supersolvable and $l_p(G) = 1$. ($l_p(G)$ stands for $p$-length of $G$.)

(2) If $p \not\equiv 1 \pmod{q}$ for every prime divisor $q$ of $|G|$, then $G$ possesses a supersolvable normal $p$-complement.

(3) If $N_G(P)$ is $p$-nilpotent, where $P$ is a Sylow $p$-subgroup of $G$, then $G$ possesses a supersolvable normal $p$-complement.

Proof. (1) First, by induction on $|G|$, we prove that $G$ is $p$-supersolvable. By [5, Lemma 4, p. 230], $G$ is $p$-solvable. We can assume $O_p'(G) = 1$. In addition, by [3, 8.6 Satz, p. 713], we can assume $\Phi(G) = 1$. Hence we have $G = LM$, where $L$ is a minimal normal elementary Abelian $p$-group and $M$ is a maximal subgroup of $G$ with $L \cap M = 1$. Let $R \in Syl_p(M)$; then $P = LR \in Syl_p(G)$. If $A$ stands for a maximal subgroup of $P$ containing $R$, then $A = R(A \cap L)$. By $T_p'$, $A = P \cap K$ with $K$ $p'$-quasi-normal in $G$. Let $Q \in Syl_q(G)$ for every $q \not\equiv p$. We have $KQ = QK \leq G$, hence $L \cap K = L \cap KQ \leq QK = QK$. It follows that $(Syl_q(G)|q \not\equiv p)$ and $N_G(L \cap K) = N_G(L \cap P \cap K) = N_G(L \cap A)$. Clearly, $G = (Syl_q(G)|q \not\equiv p) \cdot P \leq N_G(L \cap A)$. So $L \cap A = 1$, whence $|L| = p$. On the other hand, by induction, $G/L$ is $p$-supersolvable, hence $G$ is $p$-supersolvable.

Now we prove $l_p(G) = 1$. We can assume $O_p'(G) = 1$. Then, since $G$ is $p$-supersolvable, $G'$ is a $p$-group by [3, 9.1 Satz, p. 716]. From this it follows that $P \lhd G$, where $P$ is a Sylow $p$-subgroup of $G$. This implies that the $p$-length of $G$ is 1.
(2) Let $Q \in Syl_q(G)$ for each $q \in p'$, and let $V$ be any chief $p$-factor of $G$. Since $G$ is $p$-supersolvable, $|V| = p$. It follows that $Q \leq F_p(G)$ since $p \not\equiv 1 \pmod{q}$. As is well known, $F_p(G)$ is $p$-nilpotent, hence $G$ is $p$-nilpotent.

We have $G = PH$, where $P$ is a Sylow $p$-group of $G$ and $H$ is a normal $p$-complement of $G$. Since $H$ has $T_{p'}$ and $p \not\in \pi(H)$, $H$ has $T_\pi$ with $\pi = \pi(H)$. Therefore $H$ is supersolvable by (1).

(3) We can assume $O_p(G) = 1$. Thus, by (1), we have $P \triangleleft G$. Since $N_G(P)$ is $p$-nilpotent, $G = N_G(P)$ is $p$-nilpotent.

**Corollary 6** [5, Corollary 4, p. 232]. A $\pi$-group satisfying $T_\pi$ is supersolvable.

As an assertion similar to Theorem 3, we mention the following result.

**Theorem 4.** Let $G$ be a group, and let $p$ be a prime divisor of $|G|$. Suppose that each maximal subgroup of a Sylow $p$-subgroup $P$ of $G$ is $p'$-quasi-normal in $G$. If $p \not\equiv 1 \pmod{q}$ for every prime divisor $q$ of $|G|$, then $G$ is $p$-nilpotent.

**Proof.** By Sylow’s theorem, each maximal subgroup of every Sylow $p$-subgroup of $G$ is $p'$-quasi-normal in $G$. It easy to check that every quotient of $G$ satisfies the hypothesis of the theorem.

Let $Q \in Syl_q(G)$ for every $q \in p'$. If $P$ is cyclic, then $N_G(P) = C_G(P)$, and so $G$ is $p$-nilpotent by [4, Theorem 4.3, p. 252]. Hence we assume that $P$ is not cyclic. Then $P$ is the product of two maximal subgroups of it, and so $P$ is $p'$-quasi-normal in $G$. Therefore we have $PQ = QP \leq G$. The subgroup $PQ$ is solvable and $PQ$ satisfies the hypothesis of the theorem by Lemma 2. Thus, by using the argument in Theorem 3, we conclude that the subgroup $PQ$ is $p$-nilpotent. Then we have $P \leq N_G(Q)$. From this and [4, Theorem 4.5, p. 253], it follows that $G$ is $p$-nilpotent.

**Corollary 7.** Let $G$ be a group and let $p$ be the smallest prime divisor of $|G|$. If every maximal subgroup of a Sylow $p$-subgroup of $G$ is $p'$-quasi-normal in $G$, then $G$ is $p$-nilpotent.

**Corollary 8.** Let $G$ be a group, and let $p_1 < p_2 < \cdots < p_r$ be the distinct primes dividing $|G|$. If each maximal subgroup of Sylow $p_i$-subgroups of $G$ is $\{p_{i+1}, \ldots, p_r\}$-quasi-normal in $G$, $i = 1, \ldots, r - 1$, then $G$ satisfies the Sylow tower property.

**Corollary 9.** Let $G$ be a group. If each maximal subgroup of a Sylow $p$-subgroup of $G$ is $p'$-quasi-normal in $G$ for every prime divisor of $p$ of $|G|$, then $G$ is supersolvable.

**Proof.** By Corollary 8, $G$ is solvable. Thus, by using the argument in the proof (1) of Theorem 3, it follows that $G$ is supersolvable.

**Corollary 10** (Srinivasan [6]). Let $G$ be a group. If each maximal subgroup of Sylow subgroups of $G$ is $S$-quasi-normal in $G$, then $G$ is supersolvable.

**References**


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