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LIE AND JORDAN IDEALS IN $B(c_0)$ AND $B(l_\rho)$

K.-H. FÖRSTER AND B. NAGY

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ABSTRACT. It is shown that ideals with respect to the canonical Lie (commutator) product in these algebras are exactly the linear manifolds that contain the images of their elements under the action of inner automorphisms induced by invertible spectral operators of scalar type. Jordan ideals in these algebras are identical with two-sided associative ideals and are also applied to a characterization of Lie ideals.

It is well known that any associative algebra $\mathbf{A}$ becomes a Lie algebra and a Jordan algebra under the products defined by $[F, G] = FG - GF$ and $F \circ G = \frac{1}{2}(FG + GF)$ for $F, G \in \mathbf{A}$, respectively. (The ground field is $\mathbb{C}$.) A linear manifold $\mathbf{L}$ in $\mathbf{A}$ is called a Lie (Jordan) ideal if the corresponding (Lie or Jordan) product belongs to $\mathbf{L}$ for every $F \in \mathbf{A}$ and $G \in \mathbf{L}$. It is then a two-sided ideal with respect to the corresponding product. Similarly, by an associative ideal we shall understand a two-sided ideal under the associative multiplication.

Remarkable characterizations of the (not necessarily closed) Lie and Jordan ideals in the case when H is a complex infinite-dimensional separable Hilbert space and $\mathbf{A}$ is $B(H)$, the (associative) algebra of all bounded linear operators in H, were described by Fong, Miers, and Sourour [3; Theorems 1 and 3] and obtained partly by Topping (see [3, 11]). Fong and Murphy [4] showed that [3, Theorem 1] is valid in the nonseparable case, too. The purpose of this paper is to prove that [3, Theorems 1 and 3] can be extended in a suitable form for the case when the underlying complex Banach space is either c_0 or l_ρ $(1 \leq \rho < \infty)$.

The proofs are based on some fundamental results of Pełczynski [9] on complemented subspaces and isomorphisms in the Banach spaces above. Owing to them, we can make use of several ideas, applied earlier in the case of a separable Hilbert space, which we shall not reproduce here in detail. For a reference on spectral operators see, e.g., the book of Dunford and Schwartz [1].

For a bounded linear operator in a Banach space X, ker and im will denote its kernel and range subspace, respectively. An idempotent will be called any operator $P \in B(X)$ satisfying $P^2 = P$. The notations $(X \oplus X \oplus \cdots)_X$ and $\sim$

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will have the same meaning as in [9]: in particular, $X \sim Y$ means that there is a topological isomorphism (i.e., a linear homeomorphism) between the Banach spaces X and Y. The norms in different Banach spaces will be denoted by $|\cdot|$ with or without subscripts. If $\mathbf{A}$ is an associative algebra and A_1, A_2 are subsets of $\mathbf{A}$, then $[A_1, A_2]$ will denote the set of all sums of elements of the form $[\alpha_1, \alpha_2]$ where $\alpha_i \in A_i$ for $i = 1, 2$. If J is an ideal in $\mathbf{A}$, then let

$$J^{\sim} = \{a \in \mathbf{A} : [a, \alpha] \in \mathbf{J} \text{ for every } \alpha \text{ in } \mathbf{A}\}.$$

We shall need the following

Lemma 1. *Let either $X = l_p$ $(1 \le p < \infty)$ or $X = c_0$, and let $A \in B(X)$. Then A is the sum of two commutators, i.e., of operators of the form $FG - GF$ $(F, G \in B(X))$.*

Proof. Assume first that $\dim \ker A = \infty$. Making use of [9, Lemma 2], we can find an infinite-dimensional subspace Y of $\ker A$ with the properties that $Y \sim X$, Y is complemented in X, and if Y' is any complementary subspace of Y in X then $\dim Y' = \infty$. Fix any such Y'. By [9, Theorem 1], Y' is then topologically isomorphic to X. Applying [9, Proposition 3], we obtain that

$$X \sim Y' \oplus Y \sim Y' \oplus X \sim Y' \oplus (X \oplus X \oplus \cdots)_X \sim (X \oplus X \oplus \cdots)_X.$$

Denote the last direct sum space by D. We see from the above that there is a topological isomorphism $J: X \to D$ such that J^{-1} maps all but the first component spaces of D into $Y \subset \ker A$. Let $A^+ = JAJ^{-1}$ be the operator "corresponding to A" in $B(D)$. Then A^+ vanishes on all but the first component spaces of D, hence A^+ has the following infinite operator matrix representation in $(X \oplus X \oplus \cdots)_X$ (cf. Halmos [6, Problems 55 and 186]):

$$A^+ = \begin{pmatrix} A_0 & 0 & \cdots \\ A_1 & 0 & \\ A_2 & 0 & \\ \vdots & \vdots & \end{pmatrix},$$

where $A_j \in B(X)$ for $j = 0, 1, \dots$, and the sequence $\{|A_j x_0|;\ j = 0, 1, \dots\}$ belongs to X and has X-norm not greater than $|A^+|\,|x_0|$ for any x_0 in X.

Consider the following infinite operator matrices acting in the space $D = (X \oplus X \oplus \cdots)_X$:

$$P^+ = \begin{pmatrix} 0 & 0 & 0 & \\ I & 0 & 0 & \cdots \\ 0 & I & 0 & \\ 0 & 0 & I & \\ \cdots & & & \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} A_1 & -A_0 & 0 & 0 & \cdots \\ A_2 & 0 & -A_0 & 0 & \cdots \\ A_3 & 0 & 0 & -A_0 & \cdots \\ A_4 & 0 & 0 & 0 & \\ \cdots & & & & \end{pmatrix}.$$

For any $x = (x_0, x_1, \dots) \in D$, the ith component of Q^+x has X-norm not greater than $|A_{i+1}x_0| + |A_0|\,|x_{i+1}|$, hence the D-norm of Q^+x satisfies $|Q^+x|_D \le |A^+|\,|x_0| + |A_0|\,|x|_D \le (|A^+| + |A_0|)|x|_D$. Thus both operators P^+ and Q^+ belong to $B(D)$ and, as [6; Problem 186] shows, satisfy $A^+ = P^+Q^+ - Q^+P^+$. Since $A = J^{-1}A^+J$, the operator A is also a commutator.

Now let $A \in B(X)$ be arbitrary, and let K be a topological isomorphism from X onto $E = X \oplus X$. Let $A' = KAK^{-1}$. Then $A' \in B(E)$ has the

operator matrix representation

$$A' = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

The summands on the right-hand side are commutators, by the preceding paragraph. Transforming all the relevant operators into $B(X)$, we see that A is a sum of two commutators in $B(X)$.

Theorem 1. *Let* $\mathbf{L}$ *be a linear manifold in* $B(X)$, *where* X *is one of the spaces* l_p $(1 \le p < \infty)$ *or* c_0. *The following conditions are equivalent*:

(1) $\mathbf{L}$ *is a Lie ideal*;
(2) $U^{-1}\mathbf{L}U \subset \mathbf{L}$ *for every spectral operator of scalar type* U *in* $B(X)$, *whose spectrum is contained in the unit circle*;
(3) $N^{-1}\mathbf{L}N \subset \mathbf{L}$ *for every invertible spectral operator of scalar type* N *in* $B(X)$.

Proof. (3) clearly implies (2). Assume now (2), let P be an idempotent in $B(X)$ and $A \in \mathbf{L}$. The operator $U = P + i(I - P)$ is then as required in (2), $U^{-1} = P - i(I - P)$ is also of that type, and $[P, A] = PA - AP = (U^{-1}AU - UAU^{-1})/2i \in \mathbf{L}$. We shall show that each operator T in $B(X)$ is the sum of a finite number of idempotents, which will imply (1).

Let D denote $X \oplus X$. Then there is a topological isomorphism J from X onto D. It is sufficient to show that $T^+ = JTJ^{-1} \in B(D)$ is a sum of idempotents P_k^+ in $B(D)$; then T will be the sum of the idempotents $P_k = J^{-1}P_k^+J$ in $B(X)$. Let T^+ be represented by the operator matrix $\left(\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right)$ in $D = X \oplus X$. By Lemma 1, the operator $A_{11} + A_{22} - 8I$ is the sum of two commutators, say B' and C', in $B(X)$. Fixing these, let $B_{11} \in B(X)$ be arbitrary, and let

$$C_{11} = A_{11} - B_{11}, \qquad B_{22} = B' + 4I - B_{11}, \qquad C_{22} = A_{22} - B_{22}.$$

Then all these operators belong to $B(X)$ and

$$T^+ = \begin{pmatrix} B_{11} & * \\ * & B_{22} \end{pmatrix} + \begin{pmatrix} C_{11} & * \\ * & C_{22} \end{pmatrix}.$$

Here the stars denote suitable operators in $B(X)$, further $B_{11} + B_{22} - 4I = B'$ and $C_{11} + C_{22} - 4I = C'$ are commutators in $B(X)$.

A completely algebraic part of the proof of Theorem 1 in Pearcy and Topping [8] shows that any 2×2 operator matrix for which "trace" minus $4I$ is a commutator can be written as the sum of four idempotents. Hence T^+ is the sum of eight idempotents and (1) follows.

Assume now (1), and let S be an involution, i.e., a square root of the identity I in $B(X)$. For any A in $\mathbf{L}$ then we have $S^{-1}AS = A - \frac{1}{2}[S, [S, A]] \in \mathbf{L}$. Recalling that $X \sim X \oplus X \sim (X \oplus X \oplus \cdots)_X$, the following statement can essentially be proved as in Radjavi [10, Lemma 3] for the Hilbert space case: If $T = \operatorname{diag}(B, C) = \left(\begin{smallmatrix} B & 0 \\ 0 & C \end{smallmatrix}\right)$ in the direct sum $X \oplus X$ is invertible, then T is the product of six involutions.

Let N be as in (3). Applying essentially the method of the proof of [6, Problem 111], we can find an idempotent P commuting with N and such that $\dim PX = \dim(I - P)X = \infty$. By [9, Theorem 1], then $PX \sim X \sim (I - P)X$.

Making use of the statement of the preceding paragraph, N is the product of six involutions, hence (3) holds.

Theorem 2. *Let $X = c_0$ or $X = l_\rho$ $(1 \le \rho < \infty)$. A subset of $B(X)$ is a Jordan ideal if and only if it is an associative ideal.*

Proof. An associative ideal is evidently a Jordan ideal. Assume now that $\mathbf{J}$ is a Jordan ideal, $T \in \mathbf{J}$, and P is an idempotent in $B(X)$ satisfying $\dim\ker P = \dim\operatorname{im} P = \infty$. Then $PTP = 2(T\circ P)\circ P - T\circ P$ belongs to $\mathbf{J}$. By [9, Theorem 1], both $\ker P$ and $\operatorname{im} P$ are topologically isomorphic to X, so we can find a topological isomorphism $V_1\colon \ker P \to \operatorname{im} P$ of these subspaces with the inverse $V_1^{-1}\colon \operatorname{im} P \to \ker P$. Define the operators V and V' in $B(X)$ by

$$V = \begin{cases} V_1 & \text{on } \ker P, \\ 0 & \text{on } \operatorname{im} P, \end{cases} \qquad V' = \begin{cases} V_1^{-1} & \text{on } \operatorname{im} P, \\ 0 & \text{on } \ker P. \end{cases}$$

Then we clearly have the following relations:

$$VV' = P, \qquad V'V = I - P, \qquad VP = 0, \qquad PV = V, \qquad PV' = 0.$$

Since $T\circ V \in \mathbf{J}$, we obtain as in the proof of [3, Theorem 3] that $(VTP)\circ V' \in \mathbf{J}$, and hence that TP and PT belong to $\mathbf{J}$.

Assume now that X is the direct sum $F \oplus Y$ of two closed subspaces and F is finite dimensional. By [9, Theorem 1], Y is then topologically isomorphic to X. Hence there are infinite-dimensional closed subspaces W and U of Y such that $Y = W \oplus U$. Denote by $D(A, B)$ the idempotent D in $B(X)$ determined by $\operatorname{im} D = A$ and $\ker D = B$. Since F is finite dimensional, the linear manifolds $F \oplus W$ and $F \oplus U$ are closed, and

$$D(F, Y) = D(F, W \oplus U) = D(F \oplus W, U) - D(W, F \oplus U).$$

The right-hand side idempotents here satisfy $\dim\ker D = \dim\operatorname{im} D = \infty$. Applying the preceding paragraph, we obtain that PT and TP belong to $\mathbf{J}$ for any idempotent P in $B(X)$. By the proof of Theorem 1, each S in $B(X)$ is the sum of eight idempotents, hence $\mathbf{J}$ is an associative ideal.

A result of Murphy [7] together with Lemma 1 and the proof of Theorem 2 yield

Theorem 3. *Let $\mathbf{B}$ denote either $B(l_\rho)$ $(1 \le \rho < \infty)$ or $B(c_0)$ and, let $\mathbf{L}$ denote a linear manifold in $\mathbf{B}$. $\mathbf{L}$ is a Lie ideal in $\mathbf{B}$ if and only if there is an ideal $\mathbf{I}$ in $\mathbf{B}$ such that $[\mathbf{B}, \mathbf{I}] \subseteq \mathbf{L} \subseteq \mathbf{I}^\sim$.*

Proof. The proof of Theorem 2 shows that the algebra $\mathbf{B}$ has a set $E = \{e_{ij};\ i, j = 1, 2\}$ of 2×2 matrix units (for this notion see, e.g., Faith [2; pp. 133–134]): take any idempotent P in $\mathbf{B}$ satisfying $\dim\ker P = \dim\operatorname{im} P = \infty$, define V and V' in $\mathbf{B}$ as in that proof, and let

$$E = \begin{pmatrix} P & V \\ V' & I - P \end{pmatrix}.$$

Lemma 1 shows that $[\mathbf{B}, \mathbf{B}] = \mathbf{B}$. Hence [7, Theorem 5] gives the stated equivalence.

Corollary. *With the notation above $\mathbf{B}$ has no proper finite codimensional Lie ideals.*

Proof. [7, Theorem 6] and Theorem 3 show that **B** has a proper finite-codimensional Lie ideal if and only if **B** has a proper finite-codimensional ideal. Gohberg, Markus, and Feldman [5] showed that the only proper closed nonzero ideal in **B** is the ideal of compact operators, which is clearly not finite codimensional.

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DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF BERLIN, D 1000 BERLIN-12, GERMANY

E-mail address: foerster@math.tu-berlin.de

DEPARTMENT OF MATHEMATICS, FACULTY OF CHEMICAL ENGINEERING, TECHNICAL UNIVERSITY OF BUDAPEST, H-1521 BUDAPEST, HUNGARY

E-mail address: UUCP>H2939VEG@ELLA.HU