POSITIVE HARMONIC MAJORIZATION OF THE REAL PART OF A HOLOMORPHIC FUNCTION

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Abstract. Let $U$ be the unit disc. This paper investigates which domains $D$ in the complex plane have the property that $\Re f$ belongs to $h^1$, or the more restrictive property that $e^f$ belongs to the Smirnov class $N^+$, for every holomorphic function $f: U \to D$.

1. Introduction

For each domain (i.e., connected open set) $D$ in $\mathbb{C}$, let $\mathcal{H}(U, D)$ be the class of all holomorphic functions from the open unit disc $U$ into $D$. As usual, let $N$ be the Nevanlinna class of all holomorphic functions $f$ on $U$ for which

$$\sup_{0 < r < 1} \int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta < \infty.$$  

Each function $f$ in $N$ has a nontangential limit, denoted by $f(e^{i\theta})$, at almost every boundary point $e^{i\theta}$. The Smirnov class $N^+$ is the subclass of functions $f$ in $N$ for which

$$\int_0^{2\pi} \log^+ \left| f(re^{i\theta}) \right| d\theta \to \int_0^{2\pi} \log^+ \left| f(e^{i\theta}) \right| d\theta \quad (r \to 1-).$$

A discussion of the classes $N$ and $N^+$ can be found in Garnett [5, Chapter II].

Theorem A below is classical; see Helms [6, Theorem 8.33] for the “if” assertion and Frostman [4, §52] or Nevanlinna [8, VII, §4.2] for the converse. Theorem B was established more recently by Ahern and Cohn [1]. For an introduction to the notion of thin sets the reader is referred to [6, Chapter 10].

Theorem A. Let $D$ be a domain in $\mathbb{C}$. Then $f \in N$ for every $f$ in $\mathcal{H}(U, D)$ if and only if $\partial D$ has positive logarithmic capacity.

Theorem B. Let $D$ be a domain in $\mathbb{C}$. Then $f \in N^+$ for every $f$ in $\mathcal{H}(U, D)$ if and only if $\mathbb{C} \setminus D$ is nonthin at $\infty$.

In this paper we investigate which domains $D$ have the property that $e^f \in N$, or that $e^f \in N^+$, for every $f$ in $\mathcal{H}(U, D)$. We note that $e^f \in N$ if

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and only if the real part of \( f \) can be written as the difference of two positive harmonic functions on \( U \), i.e., \( \Re f \in h^1 \). Further (assuming that \( e^f \in \mathcal{N}^+ \)), \( e^f \in \mathcal{N}^+ \) if and only if \( \Re f \) is majorized in \( U \) by the Poisson integral of its nontangential boundary values.

We will denote the right half-plane by \( D_0 \). It is obviously the case that if \( D \subseteq D_0 \) then \( \Re f \in h^1 \) for every \( f \) in \( \mathcal{H}(U, D) \). The following result describes the situation for simply connected domains that contain \( D_0 \).

**Theorem 1.** Let \( D \) be a simply connected domain that contains \( D_0 \). Then \( \Re f \in h^1 \) for every \( f \) in \( \mathcal{H}(U, D) \) if and only if
\[
\int_{-\infty}^{\infty} \frac{\text{dist}(iy, \partial D)}{1 + y^2} \, dy < \infty.
\]

As will be seen in §2, Theorem 1 follows easily from a known result on the angular derivative problem. Now suppose that \( D \) is a simply connected domain that contains \( D_0 \) and satisfies (1). If \( D_1 \) is a domain (not necessarily simply connected) contained in \( D \), then clearly \( \Re f \in h^1 \) (or, equivalently, \( e^f \in \mathcal{N}^+ \)) for every \( f \) in \( \mathcal{H}(U, D_1) \). The following result identifies which of these domains have the stronger property that \( e^f \in \mathcal{N}^+ \) for every \( f \in \mathcal{H}(U, D_1) \).

**Theorem 2.** Let \( D \) be a simply connected domain that contains \( D_0 \) and satisfies (1), and let \( D_1 \) be a domain contained in \( D \). Then \( e^f \in \mathcal{N}^+ \) for every \( f \) in \( \mathcal{H}(U, D_1) \) if and only if \( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \), where
\[
D_1^* = \{(x_1, \ldots, x_4) \in \mathbb{R}^4 : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \in D_1\}.
\]

Here \( \infty \) denotes the Alexandroff point for \( \mathbb{R}^4 \). The condition "\( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \)" is equivalent to "\( D_0 \setminus D_1 \) is not minimally thin at \( \infty \) with respect to \( D_0 \)"; but the proof of Theorem 2 (see §3) does not use any results concerning minimally thin sets.

Let \( \Pi[g] \) denote the Poisson integral in \( U \) of a function \( g \) in \( L^1(\partial U) \). The following is a simple consequence of Theorem 2.

**Corollary.** Let \( D_1 \) be a domain contained in \( D_0 \). Then \( u = \Pi[u|_{\partial U}] \) for every \( f = u + iv \) in \( \mathcal{H}(U, D_1) \) if and only if \( \mathbb{R}^4 \setminus D_1^* \) is nonthin at \( \infty \).

2. Proof of Theorem 1

2.1. The following result is due to Oikawa [9], who formulated it in terms of an infinite strip rather than a half-plane. A closely related result had previously been given by Rodin and Warschawski [10, Theorem 2].

**Theorem C.** Let \( D \) be a simply connected domain that contains \( D_0 \). Then (1) holds if and only if there is a one-to-one conformal map \( g \) of \( D \) onto \( D_0 \) such that \( g(z)/z \) has a finite nonzero limit as \( |z| \to \infty \) in \( \{re^{i\theta} : |\theta| < \theta_0\} \) for each \( \theta_0 \) in \( (0, \pi/2) \).

The "if" part of Theorem 1 follows easily. To see this, let \( D \) be as in the statement of Theorem 1 and suppose (1) holds. Then there is a function \( g \) as in Theorem C. Let \( z = x + iy \), and put \( l = \lim_{x \to \infty} g(x)/x \). Clearly \( l \) is real, so \( l \in (0, \infty) \). Thus \( \text{Reg} \) is a positive harmonic function on \( D \) whose Poisson integral representation in \( D_0 \) includes the term \( lx \). Hence \( \text{Reg} \) majorizes \( lx \) on \( D_0 \). It follows that if \( f \in \mathcal{H}(U, D) \) the function \( l^{-1}\text{Reg} \circ f \) is a positive harmonic majorant of \( \Re f \), and this implies that \( \Re f \in h^1 \).
2.2. Conversely, suppose that $D$ is a simply connected domain that contains $D_0$ and that $\Re f \in h^1$ for every $f$ in $H(U, D)$. It is certainly not the case that $\Re f \in h^1$ for every holomorphic function on $U$ (see below), so $D \neq \mathbb{C}$. Thus we can choose $f$ to be a one-to-one conformal mapping of $U$ onto $D$. Let $f = u + iv$. By hypothesis, $u$ has a positive harmonic majorant, $h$ say, on $U$. Since $h \circ f^{-1}(z) \geq u \circ f^{-1}(z) = x$, the positive harmonic function $H = h \circ f^{-1}$ majorizes $x$ on $D$. We define $\phi: \mathbb{R} \to [0, \infty)$ by $\phi(y) = \text{dist}(iy, \partial D)$. If $\phi(y) > 0$ then $D$ contains the open disc of centre $iy$ and radius $\phi(y)$. Thus, applying Harnack’s inequalities, we obtain $H(iy) \geq CH(\phi(y)/2 + iy) \geq C\phi(y)/2$, where $C$ is a positive constant. Since $\int_{\{\phi(y) > 0}\} H(iy)/(1 + y^2)\,dy < \infty$, it is now clear that (1) holds.

3. Proof of Theorem 2

3.1. We recall some definitions. A positive harmonic function is called quasi-bounded if it can be expressed as the limit of an increasing sequence of bounded positive harmonic functions. Let $W$ be an open subset of $\mathbb{R}^n$ ($n \geq 2$), let $s$ be a positive superharmonic function on $W$, and let $A \subseteq W$. Then the reduced function (or réduite) of $s$ relative to $A$ in $W$ is defined to be the infimum of all positive superharmonic functions $S$ on $W$ that satisfy $S \geq s$ on $A$. A subset $A$ of $\mathbb{R}^n$ ($n \geq 3$) is said to be thin at $\infty$ if the reduced function of (the constant function) $1$ relative to $A$ in $\mathbb{R}^n$ is less than $1$ at some point of $\mathbb{R}^n$. The following lemma is an immediate consequence of Huber [7, Lemma].

Lemma A. Let $A \subseteq D_0$. The following are equivalent:

(i) the reduced function of $z \mapsto x$ relative to $A$ in $D_0$ equals $x$;

(ii) the set $A^* = \{(x_1, \ldots, x_d) : (x_1^2 + x_2^2 + x_3^2)^{1/2} + i x_4 \in A\}$ is nonthin at $\infty$.

3.2. Now let $D$ be as in the statement of Theorem 2. It follows (see §2.1) that the subharmonic function $x^+$ has a harmonic majorant in $D$. Let $h$ denote the least harmonic majorant of $x^+$ in $D$. Suppose that $\mathbb{R}^4 \setminus D_1^*$ (and hence also $(D_0 \setminus D_1)^*$) is nonthin at $\infty$. It follows from Lemma A that if $s$ is any positive superharmonic function on $D$ that majorizes $h$ on $D \setminus D_1$ so that $s(z) \geq x$ on $D_0 \setminus D_1$, then $s(z) \geq x$ on $D_0$. Hence $s$ is a superharmonic majorant of $x^+$ on $D$, and so $s \geq h$ on $D$. Thus the reduced function of $h$ relative to the set $D \setminus D_1$ in $D$ equals $h$ itself, and so (see Doob [3, I.VIII.10])

$$h(z) = \int_{D \cap \partial D_1} h(w) \, d\mu_{z, D_1}(w) = \lim_{m \to \infty} h_m(z) \quad (z \in D_1)$$

where

$$h_m(z) = \int_{D \cap \partial D_1} \min\{h(w), m\} \, d\mu_{z, D_1}(w) \quad (z \in D_1)$$

and $d\mu_{z, D_1}$ denotes harmonic measure for $D_1$ and $z$. Since $h$ is a positive harmonic function on $D_1$ that majorizes $x$ there, it follows that if $f \in H(U \setminus D_1)$ then $h \circ f$ is a positive harmonic function on $U$ that majorizes the real part of $f$. Further, if we define $u_m = h_m \circ f + \Re f - h \circ f$, then each harmonic function $u_m$ is bounded above and so is majorized on $U$ by the Poisson integral of its (nontangential) boundary values. It follows, on letting
$m \to \infty$, that $\Re f$ is majorized in $U$ by the Poisson integral of its boundary values. Thus $e^f \in N^+$ for every $f$ in $H(U, D_1)$.

3.3. To prove the converse, let $f = u + iv$, where $f: U \to D_1$ is the covering map (see Ahlfors [2, Chapter 10]). If $D_1$ is bounded, the result is trivial. If $D_1$ is unbounded then

$$\int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} g(f(e^{i\theta})) \frac{d\theta}{2\pi} = \int_{\partial D_1} g \, d\mu_{f(w), D_1} \quad (w \in U)$$

for any continuous function $g$ on $\overline{D_1} \cup \{\infty\}$. If we put $g(z) = \min\{x^+, m/|z|\}$ in (2) and let $m$ tend to infinity, it follows that

$$u^+(w) \leq \int_0^{2\pi} \frac{1 - |w|^2}{|e^{i\theta} - w|^2} u^+(e^{i\theta}) \frac{d\theta}{2\pi} = \int_{\partial D_1} x^+ \, d\mu_{f(w), D_1} (z) \quad (w \in U)$$

in view of the hypothesis that $e^f \in N^+$. Since $f(U) = D_1$, we have

$$(\Re w)^+ \leq \int_{\partial D_1} x^+ \, d\mu_{w, D_1} (z) \quad (w \in D_1).$$

It follows that $x^+$ has a quasi-bounded harmonic majorant on $D_1$ and hence on $D_1 \cap D_0$. Thus $x$ is a quasi-bounded harmonic function on $D_1 \cap D_0$, and so

$$\Re w = \int_{D_0 \cap \partial D_1} x \, d\mu_{w, D_1 \cap D_0} (z) \quad (w \in D_1 \cap D_0).$$

Hence the reduced function of $x$ relative to $D_0 \setminus D_1$ in $D_0$ equals $x$ itself. It follows from Lemma A that $\mathbb{R}^4 \setminus D_1^*$ is nonthin at $\infty$, and this completes the proof of Theorem 2.

REFERENCES


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