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# REMARKS ON LOCALIZATION AND STANDARD MODULES: THE DUALITY THEOREM ON A GENERALIZED FLAG VARIETY

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ABSTRACT. An amplification of the duality theorem of Hecht, Miličić, Schmid, and Wolf is given in the setting of a generalized flag variety. As an application, we give a different proof of the reducibility theorem for principal series by Speh and Vogan.

The purpose of this note is to sketch an amplification of the duality theorem of Hecht, Miličić, Schmid, and Wolf. The main theorem in [3] establishes a natural duality between Harish-Chandra modules constructed in two different ways: the $\mathscr{D}$-module construction on a flag variety (due to Beilinson and Bernstein) on the one hand and the cohomological induction from a Borel subalgebra (due to Zuckerman) on the other hand. This also follows from a result of Bernstein, which gives a $\mathscr{D}$-module construction of Zuckerman's functors (for the construction, see [1]). In this note we introduce a similar $\widetilde{D}$-module construction on a generalized flag variety. Following Hecht, Miličić, Schmid, and Wolf, this construction is then dual to the cohomological induction from a parabolic subalgebra (when containing a $\sigma$-stable Levi factor). As an application of this result, we give a different proof of a result by Speh and Vogan [4], which asserts the irreducibility of certain standard Zuckerman modules. This introduction is followed by four sections. The first section recalls the notion of a generalized t.d.o. on a generalized flag variety introduced in [2], from which our construction stems. The main result is stated in §2. A direct transplant of the argument of [3] is sketched in §3. The application is given in the last section. Our notation and conventions will be the same as (or parallel to) those in [3].

The author would like to thank D. Miličić for pointing out the simple but crucial Lemma 2.2 and suggesting some treatment given in §4.

## 1. THE ALGEBRA $\widetilde{\mathscr{D}}_n$

Following [3], our setting is in the context of a Harish-Chandra pair $(\mathfrak{g}, K)$, consisting of a complex semisimple Lie algebra $\mathfrak{g}$ and a connected complex

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linear algebraic group $K$, together with a morphism of algebraic groups $K \to \mathrm{Aut}(\mathfrak{g})$ such that whose differential is an injection and, by identifying the Lie algebra $\mathfrak{k}$ of $K$ with its image in $\mathfrak{g}$, $\mathfrak{t}$ is the fixed point set of an involution $\sigma$ of $\mathfrak{g}$. As usual $X$ denotes the flag variety associated to $\mathfrak{g}$. Once and for all, we fix a subset $S$ of simple roots and let $X_{\mathfrak{l}}$ be the variety of all parabolic subalgebras of type $S$. This is the space where our construction takes place. We start with recalling the construction of a generalized t.d.o. on $X_{\mathfrak{l}}$; for details see [2]. We use the pair $(\mathfrak{p}, \mathfrak{u})$ to identify $(\mathfrak{p}_x, \mathfrak{u}_x)$ for $x \in X_{\mathfrak{l}}$ ($\mathfrak{u}_x$ is the nilpotent radical of $\mathfrak{p}_x$, the parabolic subalgebra corresponding to $x$). As in the case of the flag variety, we have the sheaf of Lie algebras $\tilde{\mathfrak{g}}^0 = \mathscr{O}_{X_{\mathfrak{l}}} \otimes_{\mathbb{C}} \mathfrak{g}$. Put

$$\begin{aligned}
\tilde{\mathfrak{p}}^0 &= \{f \in \tilde{\mathfrak{g}}^0 | f(x) \in \mathfrak{p}_x \text{ for all } x \in X_{\mathfrak{l}}\},\\
\tilde{\mathfrak{u}}^0 &= \{f \in \tilde{\mathfrak{g}}^0 | f(x) \in \mathfrak{u}_x \text{ for all } x \in X_{\mathfrak{l}}\},\\
\tilde{\mathfrak{l}}^0 &= \tilde{\mathfrak{p}}^0/\tilde{\mathfrak{u}}^0.
\end{aligned}$$

Since $G$ acts on $\tilde{\mathfrak{p}}^0$ and $\tilde{\mathfrak{u}}^0$, they are sheaves of ideals of $\tilde{\mathfrak{g}}^0$. Also we have the sheaf of associative algebras $\widetilde{\mathscr{U}}^0$ generated by $\mathscr{O}_{X_{\mathfrak{l}}}$ and $\tilde{\mathfrak{g}}^0$, which is isomorphic to $\mathscr{O}_{X_{\mathfrak{l}}} \otimes_{\mathbb{C}} U(\mathfrak{g})$, as an $\mathscr{O}_{X_{\mathfrak{l}}}$-module. We define $\widetilde{\mathscr{D}}_{\mathfrak{l}} = \widetilde{\mathscr{U}}^0/\tilde{\mathfrak{u}}^0\widetilde{\mathscr{U}}^0$ and denote by $U(\tilde{\mathfrak{l}}^0)$ the sheaf of algebras generated by $\mathscr{O}_{X_{\mathfrak{l}}}$ and $\tilde{\mathfrak{l}}^0$. Let $Z(\tilde{\mathfrak{l}}^0)$ be the center of $U(\tilde{\mathfrak{l}}^0)$. We then have

1.1 **Lemma** [2, 4.2]. (1) $Z(\tilde{\mathfrak{l}}^0) = \mathscr{O}_{X_{\mathfrak{l}}} \otimes_{\mathbb{C}} Z(\mathfrak{l})$ $(\mathfrak{l} = \mathfrak{p}/\mathfrak{u})$;

(2) $Z(\mathfrak{l}) \to U(\tilde{\mathfrak{l}}^0) \to \widetilde{\mathscr{D}}_{\mathfrak{l}}$ *has its image in the center of* $\widetilde{\mathscr{D}}_{\mathfrak{l}}$.

For each $\lambda \in \mathfrak{h}^*$, let $\tilde{I}_\lambda = \mathrm{Ker}(Z(\mathfrak{l}) \to^p U(\mathfrak{h}) \to^{\chi_\lambda} C)$ where $p$ is the projection on $U(\mathfrak{h})$ via $U(\mathfrak{l}) = U(\mathfrak{h}) \oplus [(\mathfrak{l} \cap \bar{\mathfrak{n}})U(\mathfrak{l}) + U(\mathfrak{l})(\mathfrak{l} \cap \mathfrak{n})]$. In other words, $\tilde{I}_\lambda$ corresponds to the Harish-Chandra homomorphism determined by $\lambda + \rho(\mathfrak{u})$ for the triple $(\mathfrak{l}, \mathfrak{h}, R_{\mathfrak{l}}^+)$ with $R_{\mathfrak{l}}^+$ contained in the positive root system for $(\mathfrak{g}, \mathfrak{h})$ ($\rho(\mathfrak{u})$ = half sum of roots in $\mathfrak{u}$). Lemma 1.1 implies that $\tilde{I}_\lambda\widetilde{\mathscr{D}}_{\mathfrak{l}}$ is a two-sided ideal of $\widetilde{\mathscr{D}}_{\mathfrak{l}}$.

1.2 **Definition.** $\widetilde{\mathscr{D}}_\lambda = \widetilde{\mathscr{D}}_{\mathfrak{l}}/\tilde{I}_\lambda\widetilde{\mathscr{D}}_\lambda$; $\mathscr{U}_\lambda(\tilde{\mathfrak{l}}) = U(\tilde{\mathfrak{l}}^0)/\tilde{I}_\lambda U(\tilde{I}_\lambda U(\tilde{\mathfrak{l}}^0)$.

Denote by $\pi\colon X \to X_{\mathfrak{l}}$ the natural fibration. The basic property is given by

1.3 **Proposition** [2, 4.4 and 4.5]. (1) $\pi_*\mathscr{D}_\lambda \simeq \widetilde{\mathscr{D}}_\lambda$;

(2) $\mathscr{U}_\lambda(\tilde{\mathfrak{l}})$ *is the centralizer of* $\mathscr{O}_{X_{\mathfrak{l}}}$ *in* $\widetilde{\mathscr{D}}_\lambda$ *and locally* $\tilde{D}_\lambda$ *is isomorphic to* $\mathscr{U}_\lambda(\tilde{\mathfrak{l}}) \otimes_{\mathscr{O}_{X_{\mathfrak{l}}}} \mathscr{D}_{X_{\mathfrak{l}}}$ *as a* $\mathscr{O}_{X_{\mathfrak{l}}}$ *module*;

(3) $\widetilde{\mathscr{D}}_\lambda = \widetilde{\mathscr{D}}_{w\lambda}$ *for* $w$ *in the Weyl group of the pair* $(\mathfrak{l}, \mathfrak{h})$.

Let $f$ be a morphism from a smooth variety $Y$ to $X_{\mathfrak{l}}$. Put $\widetilde{\mathscr{D}}_{\lambda, Y \to X_{\mathfrak{l}}} = f^*(\widetilde{\mathscr{D}}_\lambda)$; which by definition has a right $f^{-1}\widetilde{\mathscr{D}}_\lambda$-module structure. Denote by $\widetilde{\mathscr{D}}_\lambda^f$ the sheaf of differential endomorphisms of the $\mathscr{O}_Y$-module $\widetilde{\mathscr{D}}_{\lambda, Y \to X_{\mathfrak{l}}}$ which also commute with the right $f^{-1}\widetilde{\mathscr{D}}_\lambda$-action. Because of 1.3(2), $\widetilde{\mathscr{D}}_\lambda^f$ has a similar description as $\widetilde{\mathscr{D}}_\lambda$, i.e., it is locally isomorphic to $f^*(\mathscr{U}_\lambda(\mathfrak{l})) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y$. Using these, as in Appendix A in [3], we can define the inverse image functor $f^+ (= f^*)$ and the direct image functor $f_+$ between categories $\mathscr{M}(\widetilde{\mathscr{D}}_\lambda^f)$ and $\mathscr{M}(\widetilde{\mathscr{D}}_\lambda)$. Certain

specific transformation rules of left vs right $\widetilde{\mathscr{D}}$-modules, relevant in our context, will be given in 3.1.

Taking into account the $K$-action on $X_1$, we consider the category $\mathscr{M}_{coh}(\widetilde{\mathscr{D}}_\lambda, K)$ of Harish-Chandra sheaves on $X_1$. Let $Q$ be a $K$-orbit in $X_1$ and denote by $i$ the natural immersion. Choose $x \in Q$. Let $W$ be a finitely generated $U_\lambda(\mathfrak{l}_x)$ module with a compatible algebraic $(K \cap P_x)$-action ($U_\lambda(\mathfrak{l}_x) =$ the geometric fibre of $\mathscr{U}_\lambda(\tilde{\mathfrak{l}})$ at $x$). Let $\mathscr{W}$ be the corresponding induced sheaf. Then $\mathscr{W} \in \mathscr{M}_{coh}(\widetilde{\mathscr{D}}_\lambda^i, K)$ and $R^r i_+(\mathscr{W}) \in \mathscr{M}_{coh}(\widetilde{\mathscr{D}}_\lambda, K)$ for each $r \in \mathbf{Z}$. We call these standard Harish-Chandra sheaves for the data $(Q, W)$. Note that, in view of 1.3(1), $\Gamma(X_1, \widetilde{\mathscr{D}}_\lambda) = \Gamma(X, \mathscr{D}_\lambda) = \mathscr{U}_\theta$. Therefore $H^q(X_1, R^r i_+(\mathscr{W}))$ is a Harish-Chandra module for all $q, r$ in $\mathbf{Z}$.

## 2. The main theorem

We first fix notations for standard Zuckerman modules; let $\mathfrak{p}$ be a parabolic subalgebra with a $\sigma$-stable Levi factor $\mathfrak{l}$ ($\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$) and $W$ an $(\mathfrak{l}, K \cap L)$-module, which is interpreted as a $(\mathfrak{p}, K \cap L)$-module with trivial $\mathfrak{u}$-action. The $q$th standard Zuckerman module for the data $(\mathfrak{l}, \mathfrak{u}, W)$ is defined as

$$(I^{\mathfrak{g}})^q(\mathfrak{l}, \mathfrak{u}, W) = R^q\Gamma^K_{K\cap L}(\mathrm{pro}^{\mathfrak{g}, K\cap L}_{\mathfrak{p}, K\cap L}(W)).$$

For details, see [5, 6.3].

We now describe the relevant data on the generalized flag variety $X_1$. For our purpose, we only consider those $K$-orbits in $X_1$ subject to the following restriction:

2.1 Let $x \in Q$. Then there exists a $\sigma$-stable Levi subgroup in the corresponding parabolic subgroup $P_x$.

Under this condition, the fiber $\pi^{-1}(x)$ of the natural morphism from the flag variety to $X_1$ plays the role of the flag variety for the Harish-Chandra pair $(\mathfrak{l}_x, K \cap L_x)$ with $K \cap L_x$ reductive. Unlike the situation in the flag variety, $K$-orbits in $X_1$ are generally not affinely embedded. The following simple but crucial lemma is pointed out to us by Miličić.

2.2 **Lemma.** *Under the restriction* 2.1, *$Q$ is affinely embedded in $X_1$.*

The proof in the flag variety case given in [3, 4.1, and 4.2] applies to this situation once noticing the following simple fact: If $P$ contains a $\sigma$-stable Levi factor, then the $P$-orbit of $\sigma P$ in $X_1$ is affine.

As in §2 of [3], we consider the variety $Y_1$ of ordered Levi subalgebras of type $S$ (i.e., pairs $(\mathfrak{l}, \mathfrak{u})$ such that $\mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of type $S$). $Y_1$ is then an affine variety. Given $Q$ subject to 2.1, let $\mathfrak{l}_x$ be a $\sigma$-stable Levi subalgebra of $\mathfrak{p}_x$ and $\widetilde{Q}$ the $K$-orbit in $Y_1$ passing through the pair $(\mathfrak{l}_x, \mathfrak{u}_x)$. We then have a commutative diagram

$$\begin{tikzcd} \widetilde{Q} \arrow[r, "j"] \arrow[d, "p_1"'] & Y_1 \arrow[d, "p"] \\ Q \arrow[r, "i"] & X_1 \end{tikzcd} \tag{2.3}$$

in which (a) $p_1, p, i, j$ are affine, (b) $\widetilde{Q}$, $Y_1$ are affine varieties, and (c) $p$ has fibres isomorphic to $\mathbf{C}^s$, $s = \dim$ (unipotent radical of $K \cap P_x$). Moreover,

the $K$-orbits in $Y_1$ lying above $Q$ subject to the above conditions are mutually isomorphic (called standard orbits lying above $Q$).

To describe our result, let $(Q, W)$ be data as in the end of §1, and assume $Q$ satisfies 2.1, and the unipotent radical of $K \cap P_x$ acts trivially on $W$. We fix a standard orbit $\widetilde{Q}$ lying above $Q$ and a point $y \in \widetilde{Q}$. Let $\Omega_{X_1}$ be the canonical sheaf on $X_1$ and $T_x(\Omega_{X_1})$ its geometric fibre at $x$, which is isomorphic to $\bigwedge^{\mathrm{top}} \mathfrak{u}_x$, as $(\mathfrak{l}_y, K \cap L_y)$-modules. This gives rise to the "dual" set of data $(\mathfrak{l}_y, \mathfrak{u}_y, W^\vee \otimes_{\mathbf{C}} T_x(\Omega_{X_1}))$, which then determines a family of standard Zuckerman modules.

2.4 **Theorem.** *There is a natural isomorphism of Harish-Chandra modules, for all* $q \in \mathbf{Z}$,

$$H^q(X_1, i_+(\mathscr{W}))^\vee \simeq (I^{\mathfrak{g}})^{s-q}(\mathfrak{l}_y, \mathfrak{u}_y, W^\vee \otimes_{\mathbf{C}} T_x(\Omega_{X_1})).$$

The transplant of the argument in [3] will be sketched in the next section. We remark that, by using inductions by stages, it is easy to show directly from the duality theorem of [3] that 2.4 holds when $W$ is a standard module.

## 3. Proof of the main theorem

In this section, we sketch a proof of 2.4 (which does not use specific information on $W$ along the fibre $\pi^{-1}(x)$). The point is that the argument in [3] really applies to our situation almost word for word. In the context of diagram (2.3), the idea of Hecht, Miličić, Schmid, and Wolf is to realize both sides of 2.4 on the variety $Y_1$. We will follow §§2–4 of [3] very closely. The following transformation rule of left vs right $\widetilde{\mathscr{D}}$-modules is useful.

3.1 **Lemma.** (1) $(\widetilde{\mathscr{D}}_\lambda)^0 = \widetilde{\mathscr{D}}_{-\lambda}$;

(2) $(\widetilde{\mathscr{D}}^p_\lambda)^0 = \widetilde{\mathscr{D}}^p_{-\lambda-2\rho(\mu)}$.

Here $\mathscr{D}^0$ means the opposite algebra to $\mathscr{D}$. (1) follows from the corresponding statement for $\mathscr{D}_\lambda$ on $X$ and 1.3(1). For (2), both being $G$-homogeneous, the verification comes down to adding the appropriate shift along the fibers.

Given $(Q, W)$ as before, we write $\widetilde{\mathscr{W}} = p_1^+(\mathscr{W})$, which is the induced sheaf on $\widetilde{Q}$ corresponding to $W$. Since the geometric nature of diagram (2.3) is the same as that of the corresponding diagram in the flag variety case, for the $\widetilde{\mathscr{D}}$-module side, 2.6 in [3] now reads as

3.2 **Proposition.** *There exists a first quadrant spectral sequence*

$$H^q(X_1, R^r i_+(\mathscr{W})) \Rightarrow H^{q+r-s}(\Gamma(Y_1, C_{Y_1|X_1}(R^0 j_+(\widetilde{\mathscr{W}}) \otimes_{\mathscr{O}_{Y_1}} \Omega_{Y_1|X_1}))).$$

Here $C_{Y_1|X_1}(R^0 j_+(\widetilde{\mathscr{W}}) \otimes_{\mathscr{O}_{Y_1}} \Omega_{Y_1|X_1})$ is, up to a shift in degree, the relative de Rham complex for the right $\widetilde{\mathscr{D}}^p_\lambda$-module $R^0 j_+(\widetilde{\mathscr{W}}) \otimes_{\mathscr{O}_{Y_1}} \Omega_{Y_1|X_1}$. It has a natural right $U(\mathfrak{g})$-action, which, composed with the principal antiautomorphism, gives the left $U(\mathfrak{g})$-module structure on the second term in 3.2 (cf. [3, p. 310]). Moreover (cf. [3, 2.8]).

3.3 **Lemma.** *As a left* $U(\mathfrak{g})$*-module,*

$$\begin{aligned}&\Gamma(Y_1, C_{Y_1|X_1}(R^0 j_+(\widetilde{\mathscr{W}}) \otimes_{\mathscr{O}_{Y_1}} \Omega_{Y_1|X_1}))\\ &\quad \simeq \Gamma(\widetilde{Q}, \widetilde{\mathscr{W}} \otimes_{\mathscr{O}_{\widetilde{Q}}} j^*(\Omega_{Y_1|X_1})) \otimes_{\Gamma(\widetilde{Q}, \widetilde{D}^{p\circ j}_{-\lambda})} R_{-\lambda, \widetilde{Q} \to Y_1|X_1}.\end{aligned}$$

*Notation.* $R_{\mu,\widetilde{Q}\to Y_1|X_1} = \Gamma(\widetilde{Q}, (\widetilde{D}^p_\mu)_{\widetilde{Q}\to Y_1}) \otimes_{\Gamma(Y_1,\mathscr{O}_{Y_1})} \Gamma\left(Y_1, \bigwedge \mathscr{T}_{Y_1|X_1}\right)$.

On the other hand, for a set of data $(\mathfrak{l}, \mathfrak{u}, W)$ as in §2 (here $W$ is a $(U_\lambda(\mathfrak{l}), K\cap L)$-module), let $y \in Y_1$ correspond to the data $(\mathfrak{l}, \mathfrak{u})$ and $\widetilde{Q}$ its $K$-orbit. Then the associated standard Zuckerman modules can be computed by (this corresponds to [3, 3.9]).

3.4 **Proposition.** *In the above setting,* $(I^{\mathfrak{g}})^q(\mathfrak{l}, \mathfrak{u}, W)$ *is the* $q$*th cohomology module of the complex*

$$\mathrm{Hom}_{\Gamma(\widetilde{Q},\widetilde{\mathscr{D}}^{p\circ j}_\lambda)}(R_{\lambda,\widetilde{Q}\to Y_1|X_1}, \Gamma(\widetilde{Q}, \widetilde{\mathscr{W}}))_{[K]}.$$

*Here the* $U(\mathfrak{g})$*-module structure comes from the right* $U(\mathfrak{g})$*-action on* $R_{\lambda,\widetilde{Q}\to Y_1|X_1}$.

Assuming this, replacing $W$ by $W^\vee \otimes_{\mathbf{C}} T_x(\Omega_{X_1})$ in 3.4 (also noticing that $W^\vee$ is in $\mathscr{M}(U_{-\lambda-2\rho(\mathfrak{u})}(\mathfrak{l}))$), we see that the right-hand side of 2.4 is computed as the cohomology of the complex

$$\mathrm{Hom}_{\Gamma(\widetilde{Q},\widetilde{\mathscr{D}}^{p\circ j}_{-\lambda})}(R_{-\lambda,\widetilde{Q}\to Y_1|X_1}, \Gamma(\widetilde{Q}, (\widetilde{\mathscr{W}} \otimes_{\mathscr{O}_{\widetilde{Q}}} j^*(\Omega_{Y_1|X_1}))^\vee))_{[K]}. \tag{3.5}$$

Here $(\widetilde{\mathscr{W}} \otimes_{\mathscr{O}_{\widetilde{Q}}} j^*(\Omega_{Y_1|X_1}))^\vee$ is the induced sheaf on $\widetilde{Q}$ corresponding to the contragredient to $W \otimes_{\mathbf{C}} T_y(\Omega_{Y_1|X_1})$. Since $i$ is affine (by 2.2), the spectral sequence in 3.2 collapses, together with 3.3, the natural pairing between the complexes in 3.3 and 3.5 implies 2.4 (cf. [3, pp. 320–322]).

The rest of this section consists of outlining the proof of 3.4. First of all, as in [3, 3.5], the standard Zuckerman modules $I^{\mathfrak{g}}(\mathfrak{l}, \mathfrak{u}, W)$ can be computed as the cohomology modules of the complex

$$\Gamma^K_{K\cap L}\left(\mathrm{pro}^{\mathfrak{g},K\cap L}_{\mathfrak{l},K\cap L}\left(\bigwedge \mathfrak{u}^* \otimes_{\mathbf{C}} W\right)\right). \tag{3.6}$$

As in §3 of [3], let $\psi_y$ be the functor from the category of $\Gamma(\widetilde{Q}, \widetilde{\mathscr{D}}^{p\circ j}_\lambda)$-modules to the category $\mathscr{M}(U_\lambda(\mathfrak{l}))$, obtained by applying localization followed by taking the geometric fibre at $y$. To compute $\psi_y$, we need the following version of 3.6 in [3].

3.7 **Lemma.** *Let* $\mu \in \mathfrak{h}^*$. *Then*

$$\psi_y(R^{-q}_{\mu,\widetilde{Q}\to Y_1|X_1}) = \left(U_{-\mu-2\rho(\mu)}(\mathfrak{l}) \otimes_{\mathbf{C}} \bigwedge^q \mathfrak{u}\right) \otimes_{U(\mathfrak{l})} U(\mathfrak{g})$$

*as a right* $U(\mathfrak{g})$*-module.*

Using $\psi_y$, $\widetilde{Q}$ being affine and $K$-homogeneous, we get an isomorphism (cf. [3, 3.7]), as $(\mathfrak{g}, K)$-modules, from the complex in 3.4 to the complex obtained by applying $\Gamma^K_{K\cap L}$ to the $K\cap L$ (the isotropy group at $y$) finite part of $\psi_y(\mathrm{Hom}_{\Gamma(\widetilde{Q},\widetilde{\mathscr{D}}^{p\circ j}_\lambda)}(R_{\lambda,\widetilde{Q}\to Y_1|X_1}, \Gamma(\widetilde{Q}, \widetilde{\mathscr{W}})))$; namely, by 3.7, the following complex:

$$\Gamma^K_{K\cap L}\left(\mathrm{Hom}_{U(\mathfrak{l})}\left(\left(U_{-\lambda-2\rho(\mathfrak{u})}(\mathfrak{l}) \otimes_{\mathbf{C}} \bigwedge \mathfrak{u}\right) \otimes_{U(\mathfrak{l})} U(\mathfrak{g}), W\right)_{[K\cap L]}\right). \tag{3.8}$$

Note that, $W$ being a $U_\lambda(\mathfrak{l})$-module,

$$\begin{aligned}&\mathrm{Hom}_{U(\mathfrak{l})}\left(\left(U_{-\lambda-2\rho(\mathfrak{u})}(\mathfrak{l}) \otimes_{\mathbf{C}} \bigwedge \mathfrak{u}\right) \otimes_{U(\mathfrak{l})} U(\mathfrak{g}), W\right)_{[K\cap L]}\\ &\quad = \mathrm{Hom}_{U(\mathfrak{l})}\left(U(\mathfrak{g}), \bigwedge \mathfrak{u}^* \otimes_{\mathbf{C}} W\right)_{[K\cap L]} = \mathrm{pro}^{\mathfrak{g},K\cap L}_{\mathfrak{l},K\cap L}\left(\bigwedge \mathfrak{u}^* \otimes_{\mathbf{C}} W\right).\end{aligned}$$

It then follows that standard Zuckerman modules are calculated as the cohomology of the complex (3.8), hence of the complex in 3.4.

## 4. Applications

As an application of Theorem 2.4, we have

4.1 **Theorem.** *Assume* $\mathfrak{l}$ *and* $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ *are* $\sigma$*-stable,* $W$ *is an irreducible* $(U_\lambda(\mathfrak{l}), K \cap L)$*-module, and* $\langle \lambda, \check{\alpha} \rangle \neq 1, 2, \ldots$ *for all* $\alpha \in R^+ - \langle S \rangle$. *Then*

(1) $(I^{\mathfrak{g}})^q(\mathfrak{l}, \mathfrak{u}, W \otimes_{\mathbf{C}} \bigwedge^{\dim \mathfrak{u}} \mathfrak{u}) = 0$ *for* $q \neq s$;

(2) $(I^{\mathfrak{g}})^s(\mathfrak{l}, \mathfrak{u}, W \otimes_{\mathbf{C}} \bigwedge^{\dim \mathfrak{u}} \mathfrak{u})$ *is either irreducible or zero and is irreducible if, moreover,* $\lambda$ *is regular.*

Here $\langle S \rangle = \{$the roots spanned by $S\}$. This result is contained in Speh-Vogan [4] using quite a different method. See also [6].

To start the proof, consider the diagram

$$\begin{array}{ccccc} \pi^{-1}(x) & \xrightarrow{\ l\ } & \pi^{-1}(Q) & \xrightarrow{\ k\ } & X \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \{x\} & \xrightarrow{\ j\ } & Q & \xrightarrow{\ i\ } & X_1 \end{array} \tag{4.2}$$

where $x$ corresponds to the parabolic subalgebra $\mathfrak{p}$ in 4.1 and $Q$ is the $K$-orbit of $x$ in $X_1$. Note that, $\mathfrak{p}$ being $\sigma$-stable, $Q$ is closed in $X_1$. Hence $\pi^{-1}(Q)$ is also closed in $X$. In view of 1.3(3), we may assume $\lambda$ to be antidominant w.r.t. $R^+(\mathfrak{g}, \mathfrak{h})$.

4.3 **Lemma.** *There exists an irreducible* $\mathscr{L} \in \mathscr{M}_{coh}(\mathscr{D}_\lambda^k, K)$ *such that* $\pi_* \mathscr{L} = \mathscr{W}$.

Assuming 4.3, we now prove 4.1. Since $i$ is affine, we have $R^q \pi_* k_+ \simeq i_+ R^1 \pi_*$ (cf. [2, 4.14]). Also note that, $\lambda$ being anti-dominant, the higher direct images $R^q \pi_*$ vanish. Thus

$$H^q(X_1, i_+(\mathscr{W})) \simeq H^q(X_1, i_+ \pi_*(\mathscr{L})) \simeq H^q(X_1, \pi_* k_+(\mathscr{L})) \simeq H^q(X, k_+(\mathscr{L})).$$

Since $k$ is a closed embedding, the cohomology module is trivial unless $q = 0$ and in which case, it is either irreducible or zero; if, moreover, $\lambda$ is regular, it is irreducible. Together with 2.4, this implies 4.1.

*Proof of* 4.3. The left square of diagram 4.2 gives rise to

$$\begin{array}{ccc} \mathscr{M}_{coh}(\mathscr{D}_\lambda^{k \circ l}, K \cap L) & \xleftarrow[\ l^+\ ]{} & \mathscr{M}_{coh}(\mathscr{D}_\lambda^k, K) \\ {\scriptstyle\pi_*}\downarrow & & {\scriptstyle\pi_*}\downarrow \\ \mathscr{M}_{coh}(\widetilde{\mathscr{D}}_\lambda^{i \circ j}, K \cap L) & \xleftarrow[\ j^+\ ]{} & \mathscr{M}_{coh}(\widetilde{\mathscr{D}}_\lambda^i, K) \end{array}$$

Note that, $Q$ being $K$-homogeneous, $j^+$ is an equivalence of categories. Similarly $l^+$ is an equivalence: in fact, via the $(K \cap L)$-action on $\pi^{-1}(x)$, we have $a\colon K \times \pi^{-1}(x) \to K \times_{K \cap L} \pi^{-1}(x) = \pi^{-1}(Q)$, it is easy to check that the inverse functor to $l^+$ is given by $\mathscr{A} \mapsto p_*((\mathscr{O}_K \boxtimes \mathscr{A})^{K \cap L})$; here the superscript means taking the $(K \cap L)$-invariant part of the exterior tensor product (using the right

$(K \cap L)$-action on $\mathscr{O}_K$). In particular, $l^+$ and $j^+$ are exact; therefore, the above diagram commutes (by Base Change). The lemma now follows since, $W$ being irreducible, there exists an irreducible $\mathscr{F} \in \mathscr{M}_{coh}(\mathscr{D}_\lambda^{kol}, K \cap L)$ such that $\pi_* \mathscr{F} = \mathscr{W}$. Q.E.D.

We remark that, basically the same idea, 4.1 can be derived from 2.4 and some general results on $\widetilde{\mathscr{D}}_\lambda$-modules in §4 of [2].

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