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AN ANALOGUE OF LYAPUNOV'S CRITERION FOR $(m, n-m)$-DISFOCALITY

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ABSTRACT. An integral criterion for the $(m, n-m)$-disfocality $(1 \leq m \leq n-1)$ on $[a, b]$ of a linear nth order differential equation is given. The criterion is analogous to that of Lyapunov (1947) and Nehari (1962).

1. INTRODUCTION

The equation

$$L[y] \equiv y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0, \tag{1}$$

with $a_k(x)$ real valued, continuous on $I = [a, b]$, is 'disconjugate' on I if $y(x) \equiv 0$ is its only solution having n zeros counting multiplicities on I. Nehari's criterion [7] for disconjugacy states that the equation (1) is disconjugate on I if

$$\sum_{k=1}^{n} 2^{-k}(b-a)^{k-1}\|a_k\| \leq 1 \tag{2}$$

where $\|\cdot\|$ is the L^1-norm on I. The validity of Nehari's criterion has been established by Harman in [5]. In the case $n = 2$ and $a_1(x) \equiv 0$ the criterion (2) reduces to the well-known Lyapunov's inequality

$$\|a_2\| \leq 4/(b-a), \tag{3}$$

in which the constant on the right-hand side is the best possible. This result, together with some historical notes, can be found, for instance, in [1] or [6].

More recently, disfocality of equation (1) or its special cases has received much attention as exemplified by the papers [2–4, 8] and other references contained therein. To recall the definition of disfocality let $1 < k \leq n$ and $n(1), \ldots, n(k)$ be given positive integers such that $n(1) + \cdots + n(k) = n$. Set $s(0) = 0$ and $s(j) = n(1) + \cdots + n(j)$ for $j = 1, \ldots, k$. Equation (1) is '$(n(1), \ldots, n(k))$-right disfocal' on I if $y(x) \equiv 0$ is the only solution of the BVP (1) and

$$y^{(r)}(x_i) = 0, \qquad r = s(i-1), \ldots, s(i)-1, \quad i = 1, \ldots, k. \tag{4}$$

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(Hereafter, by disfocality we mean only right disfocality.)

In this note, assuming $1 \le m \le n-1$ is fixed we present a criterion (Theorem 2.3) analogous to that of Lyapunov and Nehari for the $(m, n-m)$-disfocality of equation (1) on I. We also show (Example 3.3) that in the special case $a_k(x) \equiv 0$, $k = 1, \dots, n-1$, our criterion reduces to the inequality

$$\|a_n\| \le (n-1)!/\{C(n-2, m-1)(b-a)^{n-1}\} \tag{5}$$

with the best possible constant on the right-hand side.

2. Disfocality criterion

We use the following functions in stating the results in this section, while we denote the jth partial derivatives of a function $G(x, t)$ with respect to x and t by $G_x^{(j)}$ and $G_t^{(j)}$, respectively.

(i) $H(x, t) = \begin{cases} 0, & t < x, \\ -1, & x < t; \end{cases}$

(ii) $C(x, t; N) = \begin{cases} -(x-t)^{N-1}/(N-1)!, & x \le t, \\ 0, & t < x, \end{cases}$

for N an integer ≥ 2;

(iii) For m, N integers $(N \ge 2,\ 1 \le m \le N-1)$ $G(x, t; m, N)$ is the Green's function for the BVP

$$y^{(N)} = 0,\ y^{(i)}(x_1) = 0 = y^{(j)}(x_2), \quad i = 0, \dots, m-1,\ j = m, \dots, N-1; \tag{5N}$$

(iv) $h(k; m) = \begin{cases} 1/(k-1)!, & 1 \le k \le n-m, \\ C(k-2, m-1-n+k)/(k-1)!, & n-m+1 \le k \le n, \end{cases}$

for k, m integers $(1 \le k \le n,\ 1 \le m \le n-1)$.

Remark 2.1. It follows by induction on m and $(1-1)^{n-1} = 0$ that

$$(n-1)!h(n; m) = C(n-2, m-1) = \sum_{i=1}^{m} (-1)^{m-i} C(n-1, i-1)$$
$$= \sum_{i=m+1}^{n} (-1)^{m+i-1} C(n-1, i-1).$$

Lemma 2.2.

(i) $G(x, t; m, n) = \begin{cases} \dfrac{\sum_{i=1}^{m} (x_1-t)^{n-i}(x-x_1)^{i-1}}{(n-i)!(i-1)!}, & x_1 \le t \le x \le x_2, \\ \dfrac{-\sum_{i=m+1}^{n} (x_1-t)^{n-i}(x-x_1)^{i-1}}{(n-i)!(i-1)!}, & x_1 \le x \le t \le x_2. \end{cases}$

(ii) $G_x^{(j)} = \begin{cases} G(x, t; m-j, n-j), & 0 \le j \le m-1, \\ C(x, t; n-j), & m \le j \le n-2, \\ H(x, t), & j = n-1. \end{cases}$

(iii) $\operatorname{Sup}\{|G_x^{(j)}(x, t; m, n)| : x_1 \le t,\ x \le x_2\} = h(n-j; m)(x_2 - x_1)^{n-j-1}$, $0 \le j \le n-1$.

Proof. (i) follows from [2, (6)] or [8, Theorem 2.1].

(ii) follows from (i) by successive differentiations of G.

(iii) We first prove this result for $j = 0$. By [2, (7)] or [8, Theorem 4.1] and (i) of this lemma we have that for each fixed t,

$$\begin{aligned}(-1)^{n-m}G_x^{(l)}(x_1, t; m, n) &= 0, \qquad l = 0, \dots, m-1,\\ (-1)^{n-m}G_x^{(m)}(x, t; m, n) &> 0, \qquad x_1 \le x < t,\\ &= 0, \qquad t < x \le x_2.\end{aligned}$$

$(-1)^{n-m}G(x, t; m, n)$ as a function of x is strictly increasing on $[x_1, t]$, nondecreasing on $[t, x_2]$, and hence attains its positive maximum at $x = x_2$. Set

$$\begin{aligned}\widetilde{G}(t; m, n) &= (-1)^{n-m}G(x_2, t; m, n)\\ &= \sum_{i=1}^{m}(-1)^{m-i}(t-x_1)^{n-i}(x_2-x_1)^{i-1}/\{(n-i)!(i-1)!\},\end{aligned}$$

$x_1 \le t \le x_2$, so that

$$\begin{aligned}\widetilde{G}_t^{(l)}(x_1; m, n) &= 0, \qquad l = 0, \dots, n-m-1,\\ \widetilde{G}_t^{(n-m)}(t; m, n) &= (x_2-t)^{m-1}/(m-1)! > 0, \qquad x_1 \le t < x_2.\end{aligned}$$

Consequently $\widetilde{G}(t; m, n)$ is a strictly increasing function of t on $[x_1, x_2]$, attaining at $t = x_2$ its maximum, that is,

$$\begin{aligned}\widetilde{G}(x_2; m, n) &= \{(x_2-x_1)^{n-1}/(n-1)!\}\sum_{i=1}^{m}(-1)^{m-i}C(n-1, i-1)\\ &= (x_2-x_1)^{n-1}h(n, m) \quad \text{(by Remark 2.1)}.\end{aligned}$$

Thus (iii) is true for $j = 0$ and by virtue of (ii) is true for $j \ge 1$.

Theorem 2.3. *Assume* $1 \le m \le n-1$ *is fixed. Equation* (1) *is* $(m, n-m)$-*disfocal on* $I = [a, b]$ *if*

$$\sum_{k=1}^{n} h(k, m)(b-a)^{k-1}\|a_k\| \le 1. \tag{6}$$

Proof. Suppose there exists a solution $y(x) \not\equiv 0$ of the BVP $(5n)$ with $[x_1, x_2] \subset I$ so that we have

$$y(x) = \int_{x_2}^{x_1} G(x, t; m, n)\{a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t)\}\,dt. \tag{7}$$

Choose $\tilde{x}_j$, $x_1 \le \tilde{x}_j \le x_2$, and $\tilde{y}_j$ such that $0 < \tilde{y}_j = |y^{(j)}(\tilde{x}_j)| = \mathrm{Max}\{|y^{(j)}(x)| : x_1 \le x \le x_2\}$, $j = 0, \dots, n-1$. Denoting the L^1-norm on $[x_1, x_2]$ by $\|\cdot\|_0$ and setting $N(y) = \sum_{j=0}^{n-1}\tilde{y}_j\|a_{n-j}\|_0$, from (7) and Lemma 2.1 we obtain

$$\tilde{y}_j < h(n-j; m)(x_2-x_1)^{n-j-1}N(y), \qquad j = 0, \dots, n-1.$$

These equations in turn imply

$$N(y) < \sum_{j=0}^{n-1}\{h(n-j, m)(x_2-x_1)^{n-j-1}\|a_{n-j}\|_0\}N(y),$$

resulting in the contradiction

$$1 < \sum_{j=0}^{n-1} h(n-j, m)(x_2 - x_1)^{n-j-1} \|a_{n-j}\|_0$$
$$\leq \sum_{j=0}^{n-1} h(n-j, m)(b-a)^{n-j-1} \|a_{n-j}\| \leq 1 .$$

This completes the proof of the theorem.

3. Special case and criterion with best constant

In this section we present the example that we conjectured in the introduction.

Use of the notation given below will be convenient in the computations to follow.

Let

$$K_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad (= r \times r \text{ nilpotent matrix})$$

so that we have

$$\exp(xK_r) = \begin{pmatrix} 1 & x & x^2/2! & \cdots & x^{r-1}/(r-1)! \\ 0 & 1 & x & \cdots & x^{r-2}/(r-2)! \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} ; \tag{8}$$

$$F(x) = F_{m\times(n-m)}(x) = (x^{m-i+j}/(m-i+j)!), \qquad 1 \leq i \leq m, \ 1 \leq j \leq n-m ; \tag{9}$$

$$\tilde{d}(x) = \tilde{d}_{(n-m)\times 1}(x) = \mathrm{col}(d_1(x), \ldots, d_{(n-m)}(x))$$

where

$$d_i(x) = (-1)^{i-1} x^{n-m+2-i}[(1/(n-m+1-i)!) - (1/(n-m+2-i)!)], \qquad i = 1, \ldots, n-m ; \tag{10}$$

$$\tilde{b}(x) = \tilde{b}_{1\times(n-m)}(x) = (x^m/m!, x^{m+1}/(m+1)!, \ldots, x^{n-1}/(n-1)!) ; \tag{11}$$

$$\tilde{e}(x) = \tilde{e}_{m\times 1}(x) = \mathrm{col}(e_1(x), \ldots, e_m(x))$$

where

$$e_i(x) = (-1)^{m+i} x^{n-i+2}[(1/(n-i+1)!) - (1/(n-i+2)!)], \qquad i = 1, \ldots, m ; \tag{12}$$

and $\tilde{f} = \tilde{f}_{(n-m)\times 1} = \mathrm{col}(f_1, \ldots, f_{(n-m)})$ where

$$f_i = \begin{cases} 0, & i = 1, \ldots, n-m-1, \\ (-1)^{n-m-1}, & i = n-m. \end{cases} \tag{13}$$

The next lemma is obvious from equations (10) and (13).

Lemma 3.1. $\operatorname{Lim}_{x\to 0}\{(2/x^2)\tilde{d}(x)\} = \tilde{f}$.

The following lemma, which involves a number of computations, is needed to verify the claim that the constant appearing in the inequality (5) is the best possible.

Lemma 3.2. *For* $0 < c < 1$ *fixed, define the function* $y(x; c)$ *as follows.*

$$y(x;c) = \begin{cases} \dfrac{p_m x^m}{m!} + \cdots + \dfrac{p_{n-1}x^{n-1}}{(n-1)!}, & 0 \le x < 1-c, \\ a_0 + a_1(x-1) + \cdots + \dfrac{a_{m-1}(x-1)^{m-1}}{(m-1)!} \\ \quad + \dfrac{(-1)^{n-m}c(x-1)^n}{n!} + \dfrac{(-1)^{n-m}(x-1)^{n+1}}{(n+1)!}, & 1-c \le x \le 1, \end{cases}$$

where the $(n-m)\times 1$ *vector* $\tilde{p} = \operatorname{col}(p_m, \dots, p_{n-1})$ *and the* $m \times 1$ *vector* $\tilde{a} = \operatorname{col}(a_0, \dots, a_{m-1})$ *satisfy the following linear algebraic system of equations*:

$$\{\exp[(1-c)K_{n-m}]\}\tilde{p} = \tilde{d}(c), \tag{14}$$

$$\{\exp[-cK_m]\}\tilde{a} - \{F(1-c)\}\tilde{p} = \tilde{e}(c). \tag{15}$$

Then the function $y(x; c)$ *has the following properties.*

(A) *It is a* C^n*-function on* $[0, 1]$.

(B) $y^{(i)}(0; c) = 0 = y^{(j)}(1; c)$, $i = 0, \dots, m-1$, $j = m, \dots, n-1$.

(C) $y^{(n)}(x; c) = \begin{cases} 0, & 0 \le x < 1-c, \\ (-1)^{n-m}(c+x-1), & 1-c \le x \le 1. \end{cases}$

(D) $y(x; c)$ *is a positive strictly increasing function of* x *on* $[0, 1]$.

(E) $(-1)^{n-m+1}[y^{(n-1)}(1-c; c)/y(1-c; c)] \to 1/h(n, m)$ *as* $c \to 0$.

Proof. (B), (C) are obvious from the definition of $y(x; c)$.

(A) follows from the choice of the vectors $\tilde{a}$, $\tilde{p}$ shown in equations (14) and (15) and the equations (8)–(10) and (12).

To prove (D) note that $(-1)^{n-m}y^{(n)}(x, c)$ is positive on $(1-c, 1]$ and $(-1)^{n-m}y^{(n-1)}(1; c) = 0$ from (B) and (C). Consequently, $(-1)^{n-m}y^{(n-1)}(x)$ is negative on $[1-c, 1]$. Since $y^{(n-1)}(x; c)$ is constant on $[0, 1-c]$, it follows that $(-1)^{n-m}y^{(n-1)}(x; c)$ is of sign (-1) on $[0, 1]$ with $(-1)^{n-m}y^{(n-2)}(1; c) = 0$ from (B). This implies that $(-1)^{n-m}y^{(n-2)}(x; c)$ is of sign $(-1)^2$ on $[0, 1]$. Continuing this argument by using the conditions at $x = 1$ in (B) we obtain that $(-1)^{n-m}y^{(m)}(x; c)$ is of sign $(-1)^{n-m}$ on $[0, 1]$, that is, $y^{(m)}(x; c)$ is positive on $[0, 1]$. Now it follows from the conditions at $x = 0$ in (B) that $y(x; c)$ is strictly increasing and hence positive on $[0, 1]$. To prove (E) note that

$$\begin{aligned} y(1-c; c) &= ((1-c)^m/m!, \dots, (1-c)^{n-1}/(n-1)!)\tilde{p} \\ &= \{\tilde{b}(1-c)\}\tilde{p} = \{\tilde{b}(1-c)\}\{\exp[-(1-c)K_{n-m}]\}\tilde{d}(c) \end{aligned}$$

by (10), (11), and (14). Moreover,

$$y^{(n-1)}(1-c; c) = p_{n-1} = d_{n-m}(c) = (-1)^{n-m-1}c^2/2!,$$

by the definition of $y(x;c)$ and equations (14), (8), and (10). Consequently as $c \to 0$, by Lemma 3.1 we have

$$\begin{aligned}
&(-1)^{n-m-1}y^{(n-1)}(1-c;c)/y(1-c;c)\\
&\quad = [\tilde{b}(1-c)\{\exp(-(1-c)K_{n-m})\}(2/c^2)\tilde{d}(c)]^{-1}\\
&\quad \to [\tilde{b}(1)\{\exp(-K_{n-m})\}\tilde{f}]^{-1}\\
&\quad = [(-1)^{n-m-1}\tilde{b}(1)\,\mathrm{col}((-1)^{n-m-1}/(n-m-1)!,\dots,1)]^{-1} \quad \text{(by (8) and (13))}\\
&\quad = [(1/m!)(1/(n-m-1)!) - (1/(m+1)!)(1/(n-m-2)!)\\
&\qquad\qquad + \cdots + (-1)^{n-m-1}(1/(n-1)!)]^{-1} \quad \text{(by (11))}\\
&\quad = [\{1/(n-1)!\}\{C(n-1,m) - C(n-1,m+1)\\
&\qquad\qquad + \cdots + (-1)^{n-m-1}C(n-1,n-1)\}]^{-1}\\
&\quad = 1/h(n,m) \quad \text{(by Remark 2.1)}.
\end{aligned}$$

The proof of the lemma is complete.

Example 3.3. For each c, $0 < c < 1$, let $y(x;c)$ be as in Lemma 3.2 and define

$$q(x;c) = \begin{cases} 0, & 0 \le x < 1-c,\\ -y^{(n)}(x;c)/y(x;c), & 1-c \le x \le 1. \end{cases}$$

Note that (i) $q(x;c)$ is continuous on $[0,1]$, (ii) $(-1)^{n-m}q(x;c) \le 0$ on $[0,1]$ by (C) and (D) of Lemma 3.2, (iii) $y(x,c) \not\equiv 0$ is a solution of the BVP

$$y^{(n)} + q(x;c)y = 0, \tag{16}$$

$$y^{(i)}(0) = 0 = y^{(j)}(1), \qquad i = 0,\dots,m-1,\ j = m,\dots,n-1, \tag{17}$$

and, consequently, (iv) equation (16) is not $(m, n-m)$-disfocal on $[0,1]$. Hence by Theorem 2.3, we have

$$\begin{aligned}
1/h(n,m) < \|q(x;c)\| &= \int_{1-c}^{1} (-1)^{n-m}(y^{(n)}(x;c)/y(x;c))\,dx\\
&< (1/y(1-c;c))\int_{1-c}^{1}(-1)^{n-m}y^{(n)}(x;c)\,dx\\
&\qquad\qquad \text{(by (D) of Lemma 3.2)}\\
&< (-1)^{n-m}[y^{(n-1)}(1;c) - y^{(n-1)}(1-c;c)]/y(1-c;c)\\
&< (-1)^{n-m+1}y^{(n-1)}(1-c;c)/y(1-c;c)\\
&\qquad\qquad \text{(by (B) of Lemma 3.2)}.
\end{aligned}$$

Now by taking the limit as $c \to 0$, we obtain by Lemma 3.2(E) that

$$\|q(x;c)\| \to 1/h(n,m) = (n-1)!/C(n-2,m-1),$$

thus proving that the constant on the right-hand side of (5) is the best possible.

Remark 3.4. Since $C(n-2,m-1) = C(n-2,n-m-1)$, it follows that if the inequality (5) is satisfied then equation (1) with $a_k(x) \equiv 0$, $k = 1,\dots,n-1$, is also $(n-m, m)$-disfocal on $[a,b]$.

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