WEAKLY CONVERGENT SEQUENCE COEFFICIENT
OF PRODUCT SPACE

GUANG-LU ZHANG

(Communicated by Palle E. T. Jorgensen)

Abstract. W. L. Bynum introduced the weakly convergent sequence coefficient WCS(X) of the Banach space X as
WCS(X) = sup{M: for each weakly convergent sequence \{x_n\} in X, there is some \(y \in \text{co}(\{x_n\})\) such that \(M \cdot \limsup_{n} \|x_n - y\| \leq A(\{x_n\})\). We consider the weakly convergent sequence coefficient of the \(l_p\)-product space \(Z = (\prod_{i=1}^{n} X_i)_{l_p}\) of the finite non-Schur space \(X_1, \ldots, X_n\) and show that \(WCS(Z) = \min\{WCS(X_i): 1 \leq i \leq n\}\).

0. Introduction

Let \(X\) be a Banach space and \(\{x_n\}\) a sequence of \(X\). For \(x \in X\), set \(r(x, \{x_n\}) = \limsup_{n} \|x_n - x\|\). \(A(\{x_n\}) = \limsup_{n} \sup_{i, j \geq n} \|x_i - x_j\|\) and \(r(\{x_n\}) = \inf_{x \in \text{co}(\{x_n\})} r(x, \{x_n\})\) are called the asymptotic diameter of \(\{x_n\}\) and the Chebyshev radius of \(\{x_n\}\) relative to \(\text{co}(\{x_n\})\), respectively, where \(\text{co}(\{x_n\})\) denotes the closed convex hull of \(\{x_n\}\). If \(A(\{x_n\}) = \liminf_{n} \inf_{i, j \geq n, i \neq j} \|x_i - x_j\|\), then the sequence \(\{x_n\}\) is called the asymptotic equidistant sequence.

In order to study the normal structure of Banach spaces, Bynum [1] introduced the weakly convergent sequence coefficient \(WCS(X)\) for a reflexive Banach space \(X\) as follows:

\[
WCS(X) = \sup \left\{ M : \text{for each weakly convergent sequence } \{x_n\}, \right. \\
\left. \text{there is some } y \in \text{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n} \|x_n - y\| \leq A(\{x_n\}) \right\}.
\]

It is easy to prove that (see [2])

\[
WCS(X) = \inf \{ A(\{x_n\}) / r(\{x_n\}) : \{x_n\} \text{ a weakly but not strongly convergent sequence in } X \}.
\]

Received by the editors January 2, 1990.
1991 Mathematics Subject Classification. Primary 46B20.
Key words and phrases. Asymptotic equidistant sequence, weakly convergent sequence coefficient.

© 1993 American Mathematical Society
0002-9939/93 $1.00 + $.25 per page
calculate $WCS(X)$. We will improve expressions (1) and (2) on $WCS(X)$ and use new expressions of $WCS(X)$ to discuss the weakly convergent sequence coefficient of the $l_p$-product space of finite non-Schur Banach spaces.

In §1 we introduce the notion of the asymptotic equidistant sequence and show some results on it since we will use it to improve the expressions of $WCS(X)$. In §2 we give four new expressions of $WCS(X)$, which are better than (1) and (2). Finally, in §3, we give a result on the weakly convergent sequence coefficient of the $l_p$-product space of finite non-Schur Banach spaces.

We always denote the set of natural numbers by $\mathbb{N}$ and we say that $x_n \overset{w}{\to} x$ to denote \{x_n\} weakly converges to $x$.

1. ASYMPTOTIC EQUIDISTANT SEQUENCE

In this section we always consider $X$ to be infinite-dimensional Banach space. For a sequence \{x_n\} in $X$, we express $\lim_n \inf \{\|x_i - x_j\| : i, j > n, i \neq j\}$ by $A_1(\{x_n\})$.

Definition 1. A sequence \{x_n\} in $X$ is said to be an asymptotic equidistant sequence if $A_1(\{x_n\}) = A(\{x_n\})$.

It is clear that if \{x_n\} is either strongly convergent or equidistant (i.e., for $i \neq j$, $\|x_i - x_j\| = $ constant), then it is asymptotic equidistant.

In the following, we prove that we can choose an asymptotic equidistant subsequence from a bounded sequence.

Proposition 1. Let \{x_n\} be a bounded sequence in $X$ and

$$d = \sup \{d' : \text{there exists a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ such that } \|x_{n_k} - x_{n_{k'}}\| \geq d' \text{ when } i \neq j\}. $$

Then for every $\varepsilon > 0$ there exists a subsequence \{y_k\} of \{x_n\} such that $d - \varepsilon < \|y_i - y_j\| < d + \varepsilon$ when $i \neq j$.

Proof. For any $\varepsilon > 0$, by definition of $d$, there exists a subsequence \{x_{n_k}\} of \{x_n\} such that $\|x_{n_k} - x_{n_{k'}}\| > d - \varepsilon$ when $i \neq j$. For any subsequence \{x_{n_k}(m)\} of \{x_{n_k}\}, if all distances from its first term $x_{n_k}(m)$ to the other terms, except those that are finite, are not smaller than $d + \varepsilon$, then the $x_{n_k}(m)$ is called a deleted point.

Consider the sequence \{x_{n_k}\}. If $x_{n_k}$ is a deleted point, then delete $x_{n_k}$ and all of the finite points whose distances to $x_{n_k}$ are smaller than $d + \varepsilon$. Let the rest be \{x_{n_k}(1)\} according to the original order. If $x_{n_k}(1)$ is a deleted point also, then delete $x_{n_k}(1)$ and the associated finite points and denote the rest by \{x_{n_k}(2)\} according to the original order. If $x_{n_k}(2)$ is a deleted point, delete $x_{n_k}(2)$ and the associated finite points continuously, etc. But this process cannot proceed indefinitely, or else we get a sequence $x_{n_k}, x_{n_k}(1), x_{n_k}(2), \ldots$ of deleted points, any two points of which have distance $\geq d + \varepsilon$. This is a contradiction to the definition of $d$. So the sequence \{x_{n_k}\} must have a point which is not deleted. Note it by $x_{n_k}(1)$ and let \{x_{n_k}(1)\} be a subsequence of \{x_{n_k}\} such that $\|x_{n_k}(1) - x_{n_{k'}}(1)\| < d + \varepsilon$ for every $i > 1$.

Continue to consider the sequence $x_{n_2}(1), x_{n_2}(1), \ldots$. As mentioned above, we also can get a point $x_{n_2}(2)$ from this sequence, which is not deleted, and a subsequence $x_{n_2}(2), x_{n_3}(2), \ldots$ of this sequence such that $\|x_{n_i}(2) - x_{n_{i'}}(2)\| < d + \varepsilon$ for every $i > 2$. Continue this process for the sequence $x_{n_3}(2), x_{n_4}(2), \ldots$, etc.
etc. When this process is continued infinitely, we can finally get a subsequence 
\( \{x_{nk}(k)\}_{k=1}^{\infty} \) of \( \{x_{nk}\} \) such that \( \|x_{nk}(i) - x_{nk}(j)\| < d + \epsilon \) when \( i \neq j \). The proof of the proposition is finished by letting \( y_k = x_{nk}(k) \).

**Proposition 2.** Let \( \{x_n\} \) be a bounded sequence in \( X \). Then there exists an asymptotic equidistant subsequence \( \{y_k\} \) of \( \{x_n\} \) such that \( A_1(\{x_n\}) \leq A(\{y_k\}) \leq A(\{x_n\}) \).

**Proof.** Let \( d_1 = \sup\{d'': \text{there exists a subsequence } \{x_{nk}\} \text{ of } \{x_n\} \text{ such that } \|x_{nk} - x_{nj}\| \geq d' \text{ for } i \neq j \} \). Then \( A_1(\{x_{nk}\}) \leq d_1 \leq A(\{x_n\}) \). By Proposition 1, there exists a subsequence \( \{x_{n(1)}(1)\} \) of \( \{x_{n(1)}\} \) such that \( d_1 - 1 < \|x_{n(1)}(1) - x_{n'(1)}\| < d_1 + 1 \) for \( i \neq j \). Let \( d_2 = \sup\{d'' : \text{there exists a subsequence } \{x_{n(1)}(1)\} \text{ of } \{x_{n(1)}\} \text{ such that } \|x_{n(1)}(1) - x_{n'(1)}\| \geq d' \text{ for } i \neq j \} \). Then \( A_1(\{x_{n(1)}\}) \leq d_2 \leq d_1 \) and, by Proposition 1 also, there exists a subsequence \( \{x_{n(2)}(2)\} \) of \( \{x_{n(1)}(1)\} \) such that \( d_2 - \frac{1}{2} < \|x_{n(2)}(2) - x_{n'(2)}\| < d_2 + \frac{1}{2} \) for \( i \neq j \). Generally, if \( d_m = \sup\{d'' : \text{there exists a subsequence } \{x_{n(m-1)}(m-1)\} \text{ of } \{x_{n(m-1)}\} \text{ such that } \|x_{n(m-1)}(m-1) - x_{n'(m-1)}\| \geq d' \text{ for } i \neq j \} \), then \( A_1(\{x_{n(m-1)}\}) \leq d_m \leq d_{m-1} \) and, by Proposition 1, there exists a subsequence \( \{x_{n(m)}(m)\} \) of \( \{x_{n(m-1)}\} \) such that \( d_m - \frac{1}{m} < \|x_{n(m)}(m) - x_{n'(m)}\| < d_m + \frac{1}{m} \) for \( i \neq j \). Thus we can obtain a real sequence \( \{d_m\} \) satisfying

\[(3) \quad A_1(\{x_n(1)\}) \leq \cdots \leq d_m \leq d_{m-1} \leq \cdots \leq d_1 \leq A(\{x_n\}),\]

and a series of sequences \( \{x_{n(1)}\}, \{x_{n(2)}\}, \ldots, \{x_{n(k)}\}, \ldots \), where each sequence is a subsequence of the last one, satisfying

\[(4) \quad d_m - \frac{1}{m} < \|x_{i(m)} - x_{j(m)}\| < d_m + \frac{1}{m} \]

for each \( m \in \mathbb{N} \) and \( i \neq j \). Let \( y_k = x_{nk}(k) \). Then \( \{y_k\} \) satisfies the requirement of the proposition. Indeed, let \( d = \lim_m d_m \); it is sufficient to prove that \( A_1(\{y_k\}) = A(\{y_k\}) = d \). For every \( \epsilon > 0 \), by (4), there is \( N \in \mathbb{N} \) such that \( d - \epsilon < \frac{1}{k} < d + \epsilon \) when \( k > N \). Hence, when \( i, j > N \) and \( i \neq j \), \( d - \epsilon < \|y_i - y_j\| < d + \epsilon \), which shows that \( A_1(\{y_k\}) > d - \epsilon \) and \( A(\{y_k\}) < d + \epsilon \). By the arbitrariness of \( \epsilon \), \( A_1(\{y_k\}) = A(\{y_k\}) = d \) holds.

Recall the famous Riesz Lemma: for every \( \epsilon > 0 \), there exists a sequence \( \{x_n\} \) in \( S(X) \), the unit sphere of \( X \), such that \( \|x_i - x_j\| > 1 - \epsilon \) when \( i \neq j \). Hence, by Proposition 2, there exist asymptotic equidistant sequences \( \{x_n\} \) in \( S(X) \) such that \( A(\{x_n\}) > 1 - \epsilon \) for every \( \epsilon > 0 \). In addition, reviewing the parameter \( \overline{J}(X)^{[3,4]} \) of \( X \),

\[ \overline{J}(X) = \sup\{r > 0 : S(X) \text{ contains an infinite } r \text{-separated set}\} \]

we furthermore have

**Proposition 3.** For every \( \epsilon > 0 \), there exists an asymptotic equidistant sequence \( \{x_n\} \) in \( S(X) \) such that \( \overline{J}(X) - \epsilon \leq A(\{x_n\}) \leq \overline{J}(X) \).

**Proof.** Obvious. □

Considering a weakly convergent sequence, we have the following result.

**Proposition 4.** Let \( x_n \overset{w}{\rightarrow} z \) and \( \limsup_n \|x_n - z\| = a \). Then the sequence \( \{x_n\} \) has an asymptotic equidistant subsequence \( \{y_k\} \) such that \( A(\{y_k\}) \geq a \).

**Proof.** Through choice of subsequence, we can assume that \( \lim\|x_n - z\| = a \). Then it follows that \( A(\{y_k\}) \geq a \) for every asymptotic equidistant subsequence.
\{y_k\} of \{x_n\}. Otherwise, if \(A(\{y_k\}) < a\) for some subsequence \(\{y_k\}\) of \(\{x_n\}\), then, for sufficiently large \(n \in \mathcal{N}\), the diameter of \(\overline{\co}(\{y_k\}_n^\infty)\) is smaller than \(a\). This contradicts the fact \(z \in \overline{\co}(\{y_k\}_n^\infty)\). \(\square\)

Remark 1. For a non-Schur space \(X\) (i.e., weak and strong convergence of a sequence in \(X\) do not coincide), there exist sequences in \(S(X)\) which weakly converge to zero. By the proof of Proposition 4, for a sequence \(\{x_n\}\) in \(S(X)\) with \(x_n \stackrel{w}{\to} 0\), any of its asymptotic equidistant subsequences \(\{y_k\}\) satisfies \(A(\{y_k\}) \geq 1\).

2. Improvements of Expressions of the \(\text{WCS}(X)\)

In order to study the weakly convergent sequence coefficient of product space we first give some improvements of expressions (1) and (2) on \(\text{WCS}(X)\). It is assumed below that all Banach spaces are non-Schur spaces, and the area of spaces we discuss is extended from reflexive to this type of spaces. For a non-Schur space \(X\), \(\text{WCS}(X)\) is defined by (1) or (2) (both are equivalent) also.

Theorem 1.

\[\text{(5)} \quad \text{WCS}(X) = \sup \left\{ M : x_n \stackrel{w}{\to} u \Rightarrow M \cdot \limsup_{n} \left\| x_n - u \right\| \leq A(\{x_n\}) \right\}.\]

Proof. For convenience, let

\[\text{W}(X) = \sup \left\{ M : x_n \stackrel{w}{\to} u \Rightarrow M \cdot \limsup_{n} \left\| x_n - u \right\| \leq A(\{x_n\}) \right\}.\]

Since \(x_n \stackrel{w}{\to} u\) implies \(u \in \overline{\co}(\{x_n\})\), it is easy to see that \(\text{W}(X) \leq \text{WCS}(X)\). Now let \(x_n \stackrel{w}{\to} u\). Due to separability of \(\overline{\co}(\{x_n\})\), we can choose a subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\) by the diagonal method such that \(\lim_{k} \left\| x_{n_k} - u \right\| = \limsup_{n} \left\| x_n - u \right\|\) and there exist \(\lim_{k} \left\| x_{n_k} - y \right\|\) for all \(y \in \overline{\co}(\{x_n\})\). For each \(k \geq 1\), consider the sequence \(\{x_{n_j} : j \geq k\}\). For any \(\varepsilon > 0\), by definition of \(\text{WCS}(X)\), there exist \(y_k \in \overline{\co}(\{x_{n_j} : j \geq k\})\) such that

\[(\text{WCS}(X) - \varepsilon) \cdot \limsup_{j} \left\| x_{n_j} - y_k \right\| \leq A(\{x_{n_j} : j \geq k\}) \leq A(\{x_{n_j}\}).\]

Notice that \(y_k \stackrel{w}{\to} u\) and by weakly lower semicontinuity of the functional \(\limsup_j \left\| x_{n_j} - y \right\|\), we have

\[(\text{WCS}(X) - \varepsilon) \cdot \limsup_{n} \left\| x_n - u \right\| = (\text{WCS}(X) - \varepsilon) \cdot \liminf_{j} \left\| x_{n_j} - u \right\|\]

\[\leq (\text{WCS}(X) - \varepsilon) \cdot \limsup_{k} \liminf_{j} \left\| x_{n_j} - y_k \right\|\]

\[\leq A(\{x_n\}) \leq A(\{x_n\}).\]

This means that \(\text{WCS}(X) - \varepsilon \leq \text{W}(X)\). By the arbitrariness of \(\varepsilon\) and the discussion above, we have shown that \(\text{WCS}(X) = \text{W}(X)\). \(\square\)

Theorem 2.

\[\text{(6)} \quad \text{WCS}(X) = \inf \left\{ \frac{A(\{x_n\})}{r(u, \{x_n\})} : \{x_n\} \text{ weakly but not strongly converges to } u \right\} .\]
Proof. Suppose

$$A(X) = \inf \left\{ \frac{A(\{x_n\})}{r(u, \{x_n\})} : \{x_n\} \text{ weakly but not strongly converges to } u \right\}$$

and $x_n \mathrel{^w} u$ but $\{x_n\}$ does not strongly converge. For every $\varepsilon > 0$, by Theorem 1, $(\text{WCS}(X) - \varepsilon) \cdot \limsup_n \|x_n - u\| \leq A(\{x_n\})$, i.e.,

$$\text{WCS}(X) - \varepsilon \leq \frac{A(\{x_n\})}{r(u, \{x_n\})}.$$ 

Hence $\text{WCS}(X) - \varepsilon \leq A(X)$ and by the arbitrariness of $\varepsilon$ it follows that $\text{WCS}(X) \leq A(X)$. On the other hand, if $x_n \mathrel{^w} u$, then $A(X) \cdot \limsup_n \|x_n - u\| \leq A(\{x_n\})$, which implies that $A(X) \leq \text{WCS}(X)$. Thus $\text{WCS}(X) = A(X)$. \(\Box\)

Theorem 3.

(7) $\text{WCS}(X) = \inf \{A(\{x_n\}) : \{x_n\} \subset S(X) \text{ and } x_n \mathrel{^w} 0\}$.

Proof. Let $W_0(X) = \inf \{A(\{x_n\}) : \{x_n\} \subset S(X) \text{ and } x_n \mathrel{^w} 0\}$. By Theorem 2, it is clear that $\text{WCS}(X) \leq W_0(X)$. Conversely, for every $\varepsilon > 0$, choose a sequence $\{x_n\}$ in $X$ such that $x_n \mathrel{^w} u$, $\{x_n\}$ does not strongly converge, and $A(\{x_n\})/r(u, \{x_n\}) < \text{WCS}(X) + \varepsilon$. Let $y_n = x_n - u$. Then $y_n \mathrel{^w} 0$. Choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\lim_k \|y_{n_k}\| = \limsup_k \|y_n\| > 0$ and each $y_{n_k} \neq 0$; then $z_{n_k} = y_{n_k}/\|y_{n_k}\| \in S(X)$ and $z_{n_k} \mathrel{^w} 0$. Hence

$$W_0(X) \leq A(\{z_{n_k}\}) = \frac{A(\{y_{n_k}\})}{\lim_k \|y_{n_k}\|} \leq \frac{A(\{x_n\})}{r(u, \{x_n\})} < \text{WCS}(X) + \varepsilon.$$ 

By the arbitrariness of $\varepsilon$, it follows that $W_0(X) \leq \text{WCS}(X)$ and so $\text{WCS}(X) = W_0(X)$. \(\Box\)

From the results of Proposition 4 and Theorem 3 it follows immediately that

Theorem 4.

$$\text{WCS}(X) = \inf \{A(\{x_n\}) : \{x_n\} \text{ an asymptotic equidistant sequence in } S(X) \text{ and } x_n \mathrel{^w} 0\}.$$ (8)

It is obvious that it is better to use and calculate expressions (5)-(8) on $\text{WCS}(X)$ rather than expressions (1) and (2).

3. On PRODUCT SPACES

The $l_p$-product space $Z = (\prod_{i=1}^n X_i)_{l_p}$ of Banach spaces $X_1, \ldots, X_n$ is, by definition, the set of all $z = (x_1, \ldots, x_n)$, where $x_i \in X_i$ for each $i \leq n$, either with the norm $\|z\| = (\sum_{i=1}^n \|x_i\|^p)^{1/p}$ when $1 \leq p < \infty$, or with the norm $\|z\| = \max\{\|x_i\| : 1 \leq i \leq n\}$ when $p = \infty$. Now we use expression (8) to discuss $\text{WCS}(Z)$.

Theorem 5. Let $1 \leq p \leq \infty$, $Z = (\prod_{i=1}^n X_i)_{l_p}$, and all $X_i$ $(1 \leq i \leq n)$ be non-Schur spaces. Then $\text{WCS}(Z) = \min\{\text{WCS}(X_i) : 1 \leq i \leq n\}$.

Proof. Without loss of generality, we assume $n = 2$, $X_1 = X$, and $X_2 = Y$. In view of $Z$ having subspaces isometric and isomorphic to $X$ and $Y$, it is clear that $\text{WCS}(Z) \leq \min\{\text{WCS}(X), \text{WCS}(Y)\}$. On the other hand, suppose $\{z_n\} \subset S(Z)$, $z_n \mathrel{^w} 0$, and $\{z_n\}$ is an asymptotic equidistant sequence. Note
\[ z_n = (x_n, y_n), \text{ where } x_n \in X \text{ and } y_n \in Y. \text{ Then } x_n \overset{w}{\to} 0 \text{ and } y_n \overset{w}{\to} 0. \text{ Let } \{n_k\} \text{ be a subsequence of the natural number sequence such that both } \{x_{n_k}\} \text{ and } \{y_{n_k}\} \text{ are asymptotic equidistant, and } \lim_{k} \|x_{n_k}\| = a \text{ and } \lim_{k} \|y_{n_k}\| = b \text{ both exist. For } 1 \leq p < \infty, \text{ notice that } \|x_{n_k}\|^p + \|y_{n_k}\|^p = 1 \text{ implies } a^p + b^p = 1. \]

It is easy to prove
\[
\lim_{i, j \to \infty} \|z_{n_i} - z_{n_j}\| = \lim_{i, j \to \infty} \left( \|x_{n_i} - x_{n_j}\|^p + \|y_{n_i} - y_{n_j}\|^p \right)^{1/p} \\
\geq \left\{ a^p [\text{WCS}(X)]^p + b^p [\text{WCS}(Y)]^p \right\}^{1/p} \\
\geq \min\{\text{WCS}(X), \text{WCS}(Y)\}.
\]

For \( p = \infty \), notice that \( \max\{\|x_{n_k}\|, \|y_{n_k}\|\} = 1 \text{ implies } \max\{a, b\} = 1 \). It is easy to prove also
\[
\lim_{i, j \to \infty} \|z_{n_i} - z_{n_j}\| = \lim_{i, j \to \infty} \max_{i, j \to \infty} \left\{ \|x_{n_i} - x_{n_j}\|, \|y_{n_i} - y_{n_j}\| \right\} \\
\geq \max\{a \text{WCS}(X), b \text{WCS}(Y)\} \\
\geq \min\{\text{WCS}(X), \text{WCS}(Y)\}.
\]

Hence, \( \text{WCS}(Z) \geq \min\{\text{WCS}(X), \text{WCS}(Y)\} \) for \( 1 \leq p \leq \infty \), and the proof is complete. \( \square \)

Recalling that Maluta’s coefficient \( D(X) \) of a Banach space \( X \) is defined by [5]
\[
D(X) = \sup \left\{ \limsup_{n} d(x_{n+1}, \text{co} \{x_{j}\}^n) \bigg/ \text{diam} \{x_{n}\} : x_{n} \text{ a bounded nonconstant sequence in } X \right\}
\]
and \( \text{WCS}(X) = 1/D(X) \) for reflexive space \( X \) [6], we obtain immediately

**Corollary 1.** Let \( 1 \leq p \leq \infty \), \( Z = (\prod_{i=1}^{n} X_i)_{l_p} \), and all \( X_i \) (\( 1 \leq i \leq n \)) be reflexive Banach spaces. Then
\[
D(Z) = \max\{D(X_i) : 1 \leq i \leq n\}.
\]

**Remark 2.** When Landes discussed permanence properties of normal structures under a finite product of Banach spaces, he showed that [7] if both \( X \) and \( Y \) are reflexive spaces and \( \text{WCS}(X), \text{WCS}(Y) > 1 \), then \( Z = (X \times Y)_{l_1} \) has the normal structure. Applying our Theorem 5, in fact, we can obtain \( \text{WCS}(Z) > 1 \), which implies that \( Z \) has the normal structure.

**References**


Department of Mathematics & Physics, University of Petroleum, Dongying City, Shandong, People's Republic of China