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INTEGER-VALUED POLYNOMIALS ON A SUBSET

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ABSTRACT. We let D be a local (noetherian) one-dimensional unibranched domain, K its quotient field, $\mathfrak{m}$ its maximal ideal, D' its integral closure, and $\mathfrak{m}'$ the maximal ideal of D'. If E is a subset of K, we let $\operatorname{Int}(E, D)$ be the set of *integer-valued polynomials* on E, thus $\operatorname{Int}(E, D) = \{f \in K[X] | f(E) \subset D\}$. For a *fractional subset* E of D (i.e., there is a nonzero element d of D such that $dE \subset D$), we show that the prime ideals of $\operatorname{Int}(E, D)$ above $\mathfrak{m}$ are in one-to-one correspondence with the elements of the topological closure of E in the completion of K for the $\mathfrak{m}'$-adic topology.

INTRODUCTION

Throughout this paper, D is a domain; we denote by K its quotient field and by D' the integral closure of D. If E is a subset of K, we let $\operatorname{Int}(E, D)$ be the set of *integer-valued polynomials* on E, thus $\operatorname{Int}(E, D) = \{f \in K[X] | f(E) \subset D\}$; it is clearly a ring containing D, throughout, we assume that E is nonempty and D is not a field, hence that $\operatorname{Int}(E, D)$ is strictly contained in $K[X]$ (as indeed $\operatorname{Int}(E, D) \cap K = D$). If E is D itself, we simply write $\operatorname{Int}(D)$ for $\operatorname{Int}(D, D)$.

The very classical case is that of *The Ring* of integer-valued polynomials, i.e., $\operatorname{Int}(\mathbb{Z})$, or even $\operatorname{Int}(D)$ where D is the ring of integers of a number field [22, 23] and, even more generally, where D is Dedekind [5]; by localization, D turns into a discrete rank-one valuation domain and the prime ideals of $\operatorname{Int}(D)$ above the maximal ideal $\mathfrak{m}$ of D are then known to be in one-to-one correspondence with the elements of the completion $\widehat{D}$ of D: to any element α of $\widehat{D}$ corresponds the prime $\mathfrak{M}_\alpha = \{f \in \operatorname{Int}(D) | f(\alpha) \in \widehat{\mathfrak{m}}\}$ [12, 4, 13], considering the $\mathfrak{m}$-adic topology, this result generalizes to the case where D is noetherian, local, one-dimensional, with finite residue field, and is *analytically irreducible* (that is where $\widehat{D}$ is a domain) [11]. Dropping this last hypothesis, however, only two facts were known in the previous decade:

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(1) The prime of $\mathrm{Int}(D)$ above $\mathfrak{m}$ are of the type $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(D) | f(\alpha) \in \widehat{\mathfrak{m}}\}$ [13].
(2) If D *is not* analytically irreducible, then some of those primes are equal; namely, if $(\alpha - \beta)$ is a zero divisor in $\widehat{D}$, then $\mathfrak{M}_\alpha = \mathfrak{M}_\beta$ [14].

Letting D be noetherian, one-dimensional, and local, we say that D is *unibranched* if, moreover, D' is local (hence a discrete rank-one valuation domain); recall that D is analytically irreducible if and only if it is unibranched and D' is a finite D-module [3, 21]. Very recently it has been shown that if D *is not* unibranched (first in the case where D' is a finite D-module [9] and then without this hypothesis [19]), there are only finitely many primes above $\mathfrak{m}$ (namely, $\mathfrak{M}_\alpha = \mathfrak{M}_\beta$ if $(\alpha - \beta)$ lies in some nontrivial ideal of D); hence the unibranched (and nonanalytically irreducible) case was the only one to remain open. So, letting D be a unibranched domain (with finite residue field), we determine here entirely the spectrum of $\mathrm{Int}(E, D)$ for any *fractional* subset E of D (i.e., such that $dE \subset D$ for a nonzero element d of D), recovering the spectrum of $\mathrm{Int}(D)$ (i.e., $E = D$) as well as the classical results on a discrete rank-one valuation domain (i.e., $D = D'$) as special cases.

In the first section we state some generalities on integer-valued polynomials on a subset, mostly on Krull dimension and localization. In §2, generalizing a result of Gilmer, Heinzer, and Lantz [19], we show that if D is noetherian, one-dimensional, local, with finite residue field, then, for any fractional subset E of D, $\mathrm{Int}(E, D')$ is the integral closure of $\mathrm{Int}(E, D)$ and that if, moreover, D is unibranched then $\mathrm{Int}(E, D')$ is a *radical* extension of $\mathrm{Int}(E, D)$ (i.e., every element of $\mathrm{Int}(E, D')$ has a power in $\mathrm{Int}(E, D)$). In §3 we first determine the spectrum of $\mathrm{Int}(E, D)$ for a fractional subset E of a discrete rank-one valuation domain and then of a unibranched domain D (whose integral closure D' is a discrete rank-one valuation domain) as follows: Letting $\mathfrak{m}'$ be the maximal ideal of D', we consider the $\mathfrak{m}'$-adic topology, hence the topology defined by the valuation of D', rather than the $\mathfrak{m}$-adic topology, treating D as a subset of D'; we can thus deduce the spectrum of $\mathrm{Int}(D)$ from that of $\mathrm{Int}(D, D')$ and more generally establish that for any *fractional* subset E of D the primes of $\mathrm{Int}(E, D)$ above $\mathfrak{m}$ are in one-to-one correspondence with the elements of the topological closure of E in the completion of K for the topology defined by the valuation of D'. Lastly we discuss the noetherian property of $\mathrm{Int}(E, D)$, generalizing and shedding some light on the results of [19] obtained in the particular case of $\mathrm{Int}(D)$.

1. Krull dimension and localization

We start with some elementary inclusions.

Proposition 1.1. *If D and B are two domains of quotient field K such that $D \subset B$ and E, F two subsets of K and such that $E \subset F$, then* $\mathrm{Int}(F, D) \subset \mathrm{Int}(E, B)$.

If E is not contained in D then the polynomial X is not in $\mathrm{Int}(E, D)$, thus:

Proposition 1.2. *Let D be a domain and E a subset of K. Then the following statements are equivalent*:

(i) $E \subset D$.

(ii) $\mathrm{Int}(D) \subset \mathrm{Int}(E, D)$.
(iii) $D[X] \subset \mathrm{Int}(E, D)$.

It is easy to describe some primes of $\mathrm{Int}(E, D)$: If $\alpha \in E$ and $\mathfrak{p}$ is a prime ideal of D, we note $\mathfrak{P}(\alpha, \mathfrak{p}) = \{f \in \mathrm{Int}(E, D) | f(\alpha) \in \mathfrak{p}\}$ the set of polynomials of $\mathrm{Int}(E, D)$ taking on the value of α in $\mathfrak{p}$; this set is clearly a prime ideal of $\mathrm{Int}(E, D)$ above $\mathfrak{p}$ and the quotient $\mathrm{Int}(E, D)/\mathfrak{P}(\alpha, \mathfrak{p})$ is isomorphic to $D/\mathfrak{p}$; in particular, if $\mathfrak{m}$ is a maximal ideal of D then $\mathfrak{M}_\alpha = \mathfrak{P}(\alpha, \mathfrak{m})$ is a maximal ideal of $\mathrm{Int}(E, D)$. If $\mathfrak{p} \subsetneq \mathfrak{q}$ then $\mathfrak{P}(\alpha, \mathfrak{p}) \subsetneq \mathfrak{P}(\alpha, \mathfrak{q})$, and we can generalize here to $\mathrm{Int}(E, D)$ the results known for the Krull dimension of $\mathrm{Int}(D)$ [5, 13] (we denote by $\dim R$ the Krull dimension of a ring R).

First note that if E is too large it may happen that every polynomial in $\mathrm{Int}(E, D)$ is constant (i.e., $\mathrm{Int}(E, D) = D$). Also, following McQuillan [20], we shall say that E is a *fractional subset of* D if there exists a nonzero element d of D such that $dE \subset D$.

Proposition 1.3. *Let D be a domain and E a subset of K. Then* $\dim \mathrm{Int}(E, D) \geq \dim D$; *if, moreover, E is a fractional subset of D then* $\dim \mathrm{Int}(E, D) \geq \dim D + 1$.

Proof. Let $(0) = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ be a chain of primes of D; it gives rise to the chain of primes $\mathfrak{P}(\alpha, \mathfrak{p}_0) \subsetneq \mathfrak{P}(\alpha, \mathfrak{p}_1) \subsetneq \cdots \subsetneq \mathfrak{P}(\alpha, \mathfrak{p}_d)$ of $\mathrm{Int}(E, D)$. Moreover, if d is such that $dE \subset D$ then $d(X - \alpha) \in \mathfrak{P}(\alpha, \mathfrak{p}_0)$, hence $(0) \subsetneq \mathfrak{P}(\alpha, \mathfrak{p}_0)$. □

If R is a domain, recall that its *valuative dimension* $\dim_v R$ is the supremum of the Krull dimensions of the overrings of R in the quotient field of R; Also recall that a *Jaffard domain* is a domain such that $\dim_v R = \dim R$; and lastly recall that a noetherian domain is a Jaffard domain [1].

Corollary 1.4. *Let D be a domain and E a subset of K. Then*

(i) $\dim \mathrm{Int}(E, D) \leq \dim_v D + 1$;
(ii) *if D is a Jaffard domain then* $\dim \mathrm{Int}(E, D) \leq \dim D + 1$;
(iii) *if D is a Jaffard domain and E a fractional subset of D, then* $\dim \mathrm{Int}(E, D) = \dim D + 1$.

Proof. (i) $\mathrm{Int}(E, D)$ satisfies the inclusions $D \subset \mathrm{Int}(E, D) \subset K[X]$, and the result follows from [2, Lemma 1.1]; (ii) and (iii) are clear. □

We conclude this section with localization properties:

Proposition 1.5. *Let D be a domain, E a subset of K, and S a multiplicative subset of D. Then*

(i) $S^{-1}\mathrm{Int}(E, D) \subset \mathrm{Int}(E, S^{-1}D)$;
(ii) *if E is a subset of D and D is noetherian, then* $S^{-1}\mathrm{Int}(E, D) = \mathrm{Int}(E, S^{-1}D)$;
(iii) *if $A = E$ is a subring of D and S is a multiplicative subset of A, then* $S^{-1}\mathrm{Int}(A, D) \subset \mathrm{Int}(S^{-1}A, S^{-1}D)$. *Moreover, if A or D is noetherian, then* $S^{-1}\mathrm{Int}(A, D) = \mathrm{Int}(S^{-1}A, S^{-1}D) = \mathrm{Int}(A, S^{-1}D)$.

Proof. (i) Immediate.

(ii) Let $f \in \mathrm{Int}(E, S^{-1}D)$. The D-module generated by $f(E)$ is (included in) a finite D-module (generated by the coefficients of f); hence there exists $s \in S$ such that $sf(E) \subset S^{-1}D$, thus $f \in S^{-1}\mathrm{Int}(E, D)$.

(iii) If $f \in \mathrm{Int}(A, D)$, f is by definition a polynomial with coefficients in $K[X]$ whose homomorphic image in $(K/D)[X]$ is null on A, then its image in $S^{-1}(K/D)[X]$ is null on $S^{-1}A$ [10, Proposition 4], thus $\mathrm{Int}(A, D) \subset \mathrm{Int}(S^{-1}A, S^{-1}D)$ and the following inclusions always hold: $S^{-1}\mathrm{Int}(A, D) \subset \mathrm{Int}(S^{-1}A, S^{-1}D) \subset \mathrm{Int}(A, S^{-1}D)$. Moreover, if D is noetherian it follows from (ii) that $\mathrm{Int}(A, S^{-1}D) \subset S^{-1}\mathrm{Int}(A, D)$; the same holds if A is noetherian, with the same proof, considering A-modules instead of D-modules. □

Remark 1.6. If E is simply a subset of K (and not of D), then $f(E)$ is not necessarily included in the D-module generated by the coefficients of f and the conclusion of (ii) may fail. Indeed if, for instance, $D = \mathbb{Z}$, S is any nontrivial multiplicative set, and $E = S^{-1}\mathbb{Z}$, then $\mathrm{Int}(E, \mathbb{Z}) = \mathbb{Z}$ [20, Lemma 2.0], whereas $\mathrm{Int}(E, S^{-1}\mathbb{Z}) = \mathrm{Int}(S^{-1}\mathbb{Z})$ does contain nonconstant polynomials.

2. Integral closure

We start with a general condition for $\mathrm{Int}(E, D)$ to be integrally closed.

Proposition 2.1. *Let D be a domain and E a subset of K. Then* $\mathrm{Int}(E, D)$ *is integrally closed if and only if D is integrally closed.*

Proof. • If D is not integrally closed there exists $x \in K, x \notin D$, which is integral over D, then $x \notin \mathrm{Int}(E, D)$ but x is integral over $\mathrm{Int}(E, D)$.

• If D is integrally closed, and if $f \in K(X)$ is integral over $\mathrm{Int}(E, D)$, then $f \in K[X]$ (since f is integral over $K[X]$, which is integrally closed), and, for every $x \in E$, $f(x)$ is integral over D; hence $f(x) \in D$, thus $f \in \mathrm{Int}(E, D)$. □

In [19] it is proved that if D is a one-dimensional noetherian domain then the elements of $\mathrm{Int}(D')$ are integral over $\mathrm{Int}(D)$; this is done by assuming (without loss of generality) that D is local. Here we state a similar result for $\mathrm{Int}(E, D)$, but first we present the following

Lemma 2.2. *Let D be a noetherian domain, E a fractional subset of D, and $f \in \mathrm{Int}(E, D')$. Then $f(E)$ is contained in a D-algebra R, finitely generated as a D-module and such that $D \subset R \subset D'$.*

Proof. Let d be a nonzero element such that $dE \subset D$, and let $f = a_0/b + (a_1/b)X + \cdots + (a_n/b)X^n$, where $a_0, a_1, \ldots, a_n$ and b are in D. If $x \in E$ it is clear that $bd^n f(x) \in D$, hence $f(E)$ is included in the D-module $M = D' \cap (1/bd^n)D$. Since D is noetherian, M has finitely many generators $x_1, \ldots, x_r$ and $f(E)$ is included in the D-algebra $R = D[x_1, \ldots, x_r]$, which is also a finitely generated D-module. □

Proposition 2.3. *Let D be a one-dimensional noetherian local domain with finite residue field, and let E be a fractional subset of D. Then* $\mathrm{Int}(E, D')$ *is the integral closure of* $\mathrm{Int}(E, D)$.

Proof. Since $\mathrm{Int}(E, D')$ is integrally closed, it remains to prove that it is an integral extension of $\mathrm{Int}(E, D)$: let $f \in \mathrm{Int}(E, D')$, $f(E)$ is contained in a

D-algebra R, finitely generated as a D-module such that $D \subset R \subset D'$. If J is the Jacobson radical of R then J is the intersection of a finite number of maximal ideals with finite residue field; hence there exists an integer q such that, for every $x \in R$, $(x^q - x) \in J$. If C is the (nonzero) conductor $[D : R]$ and n is an integer such that $J^n \subset C$, then for every x in R, $(x^q - x)^n \in D$; hence $(f^q - f)^n = h \in \mathrm{Int}(E, D)$ and f is a zero of the monic polynomial $(Y^q - Y)^n - h$ with coefficients in $\mathrm{Int}(E, D)$. □

Remark 2.4. For *any* noetherian domain D (whatever its dimension), $\mathrm{Int}(D, D')$ is the integral closure of $\mathrm{Int}(D)$ [19, Proposition 2.2]; once again the argument is local and can be generalized to show that $\mathrm{Int}(E, D')$ is the integral closure of $\mathrm{Int}(E, D)$ for *many* subsets E of K (for examples subsets of D such that, for every maximal ideal $\mathfrak{m}$ of D with infinite residue field, E contains an infinite number of classes module $\mathfrak{m}$), but the question remains open in full generality. We can at least state the following.

Proposition 2.5. *Let D be a noetherian domain and E a fractional subset of D. Then* $\mathrm{Int}(E, D')$ *is almost integral over* $\mathrm{Int}(E, D)$.

Proof. Let $f \in \mathrm{Int}(E, D')$, $f(E)$ is contained in a D-algebra R, finitely generated as a D-module such that $D \subset R \subset D'$. Then, for every integer n, $f^n(E) \subset R$, and hence, for d in the (nonzero) conductor $[D : R]$, $df^n \in \mathrm{Int}(E, D)$. □

Proposition 2.3 can be improved in the case where D is unibranched (generalizing a result implicitly contained in the proof of Theorem 3.1 of [19], with the same argument):

Proposition 2.6. *Let D be a (one-dimensional noetherian local) unibranched domain with finite residue field, and let E be a fractional subset of D. Then* $\mathrm{Int}(E, D')$ *is a radical extension of* $\mathrm{Int}(E, D)$: *for any* $f \in \mathrm{Int}(E, D')$ *there is an integer n such that* $f^n \in \mathrm{Int}(E, D)$.

Proof. As above, let $f \in \mathrm{Int}(E, D')$, $f(E)$ is contained in a D-algebra R, finitely generated as a D-module such that $D \subset R \subset D'$. Since in this case D is unibranched then R is local, say with maximal ideal $\mathfrak{n}$ and some power $\mathfrak{n}^n$ of $\mathfrak{n}$ is contained in the (nonzero) conductor $[D : R]$, if q is the cardinal of $R/\mathfrak{n}$ then, for every x in R, $x^{q-1} = \varepsilon + y$ (where ε is 0 or 1 and $y \in \mathfrak{n}$); if p is the characteristic of $R/\mathfrak{n}$, it follows from a basic property of binomial coefficients that $(x^{q-1})^p = \varepsilon + y$, where ε is 0 or 1 and $y \in \mathfrak{n}^2$; by induction, letting $n = (q-1)p^{m-1}$, we have $x^n = \varepsilon + y$, where $y \in \mathfrak{n}^m$, hence $x^n \in D$, thus $f^n \in \mathrm{Int}(E, D)$. □

3. Integer-valued polynomials over a unibranched domain

If D is a local domain of maximal ideal $\mathfrak{m}$ with finite residue field and E is a subset of its quotient field K, we let $\mathcal{J} = \{f \in \mathrm{Int}(E, D) | f(E) \subset \mathfrak{m}\}$; $\mathcal{J}$ is clearly an ideal of $\mathrm{Int}(E, D)$. We first establish the following (generalizing [9]):

Lemma 3.1. *Let D be a local domain with maximal ideal $\mathfrak{m}$ and finite residue field, E a subset of K, and $\mathfrak{M}$ a prime ideal of* $\mathrm{Int}(E, D)$ *containing the ideal*

$\mathscr{I} = \{f \in \mathrm{Int}(E, D) | f(E) \subset \mathfrak{m}\}$. *Then* $\mathfrak{M}$ *is maximal, it is above* $\mathfrak{m}$, *and its residue field is isomorphic to* $D/\mathfrak{m}$.

Proof. $\mathfrak{M}$ is clearly above $\mathfrak{m}$, and if q denotes the cardinal of $D/\mathfrak{m}$ then, for any f in $\mathrm{Int}(E, D)$, $(f^q - f) \in \mathscr{I}$; hence the residue field of $\mathfrak{M}$ is of cardinal q. Therefore, it is isomorphic to $D/\mathfrak{m}$ and $\mathfrak{M}$ is maximal. □

Now if V is a discrete rank-one valuation domain with finite residue field we generalize to $\mathrm{Int}(E, V)$ the classical results on the spectrum of $\mathrm{Int}(V)$ [13]. In a first step we suppose E to be a subset *contained in* V:

Lemma 3.2. *Let* E *be a subset of a rank-one discrete valuation domain* V *with maximal ideal* $\mathfrak{n}$ *and with finite residue field. Then the primes of* $\mathrm{Int}(E, V)$ *above* $\mathfrak{n}$ *are in one-to-one correspondence with the elements of the topological closure* $\overline{E}$ *of* E *in* $\widehat{V}$: *to any element* α *of* $\overline{E}$ *corresponds the prime* $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, V) | f(\alpha) \in \hat{\mathfrak{n}}\}$.

Proof. This is an easy repetition of the classical proof [13]. $\mathfrak{n}$ is a principal ideal generated by an element π, hence $\mathscr{I} = \{f \in \mathrm{Int}(E, V) | f(E) \subset \mathfrak{n}\} = \pi\,\mathrm{Int}(E, V)$. Now every integer-valued polynomial (on E) is a continuous function on the topological closure $\overline{E}$ of E with values in the completion $\widehat{V}$ of V, thus

$$\mathrm{Int}(V) \subset \mathrm{Int}(E, V) \subset \mathscr{C}(\overline{E}, \widehat{V}).$$

The ideal $\mathscr{I}$ is the intersection $\mathrm{Int}(E, V) \cap \mathscr{C}(\overline{E}, \hat{\mathfrak{n}})$, thus

$$\mathrm{Int}(E, V)/\mathscr{I} \subset \mathscr{C}(\overline{E}, \widehat{V})/\mathscr{C}(\overline{E}, \hat{\mathfrak{n}}) = \mathscr{C}(\overline{E}, k)$$

(where $\mathscr{C}(\overline{E}, k)$ is the ring of locally constant functions on $\overline{E}$ with values in $\widehat{V}/\hat{\mathfrak{n}} = k$). Since $\overline{E}$ is compact (being a subspace of $\widehat{V}$), the primes of $\mathscr{C}(\overline{E}, k)$ are in one-to-one correspondence with the elements of $\overline{E}$: to any element x corresponds the set of functions null at x [3, II, §4, exercise 17].

Every prime of $\mathrm{Int}(E, V)$ above $\mathfrak{n}$ contains π, hence it contains $\mathscr{I}$ and, therefore, it is maximal (Lemma 2.1); hence every prime of $\mathrm{Int}(E, V)/\mathscr{I}$ lifts in $\mathscr{C}(\overline{E}, k)$ (therefore, every prime of $\mathrm{Int}(E, V)$ lifts in $\mathscr{C}(\overline{E}, \widehat{V})$). In conclusion the primes of $\mathrm{Int}(E, V)$ containing $\mathfrak{n}$ are of the type

$$\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, V) | f(\alpha) \in \hat{\mathfrak{n}}\}, \quad \text{where } \alpha \in \overline{E}.$$

Lastly, if $\alpha \neq \beta$, there is f in $\mathrm{Int}(V)$ such that $f(\alpha) \in \hat{\mathfrak{n}}$ but $f(\beta) \notin \hat{\mathfrak{n}}$ (because $\mathrm{Int}(V)$ is dense in $\mathscr{C}(\widehat{V}, \widehat{V})$ [11, 13]), hence $\mathfrak{M}_\alpha \neq \mathfrak{M}_\beta$, since $\mathrm{Int}(V) \subset \mathrm{Int}(E, V)$. □

The previous lemma generalizes to a fractional subset:

Proposition 3.3. *Let* E *be a fractional subset of a rank-one discrete valuation domain* V *with maximal ideal* $\mathfrak{n}$ *and finite residue field. Then the primes of* $\mathrm{Int}(E, V)$ *above* $\mathfrak{n}$ *are in one-to-one correspondence with the elements of the topological closure* $\overline{E}$ *of* E *in the completion* $\widehat{K}$ *of the quotient field* K *of* V: *to any element* α *of* $\overline{E}$ *corresponds the prime* $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, V) | f(\alpha) \in \hat{\mathfrak{n}}\}$.

Proof. Since E is a fractional subset of V, there exists an element d of V such that $dE \subset V$; we let $F = dE$ and ϕ be the ring homomorphism such that $\phi(f(X)) = f(X/d)$, ϕ is clearly an isomorphism from $\mathrm{Int}(E, V)$ onto

$\mathrm{Int}(F, V)$. Now F is a subset of V and from Lemma 3.2 the primes of $\mathrm{Int}(F, V)$ above $\mathfrak{n}$ are in one-to-one correspondence with the elements of the topological closure $\overline{F}$ of F in the completion $\widehat{V}$ of V (contained in the completion $\widehat{K}$ of K). If $\overline{E}$ is the topological closure of E in $\widehat{K}$ then $\overline{F} = d\overline{E}$. Let α be an element of $\overline{F}$, $\beta = \alpha/d$, and $\mathfrak{N}_\alpha = \{f \in \mathrm{Int}(F, V) | f(\alpha) \in \hat{\mathfrak{n}}\}$. Then $\phi^{-1}(\mathfrak{N}_\alpha) = \mathfrak{M}_\beta = \{f \in \mathrm{Int}(E, V) | f(\beta) \in \hat{\mathfrak{n}}\}$. □

We can generalize further to a unibranched domain:

Theorem 3.4. *Let D be a (one-dimensional noetherian local) unibranched domain with maximal ideal $\mathfrak{m}$ and finite residue field. Let $\mathfrak{m}'$ be the maximal ideal of D' and E be a fractional subset of D. The primes of* $\mathrm{Int}(E, D)$ *above $\mathfrak{m}$ are in one-to-one correspondence with the elements of the topological closure $\overline{E}$ of E in the completion $\widehat{K}$ of K for the topology defined by the valuation of D': to any element α of $\overline{E}$ corresponds the prime $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, D) | f(\alpha) \in \hat{\mathfrak{m}}'\}$.*

Proof. We consider the inclusion $\mathrm{Int}(E, D) \subset \mathrm{Int}(E, D')$. Since $\mathrm{Int}(E, D')$ is integral over $\mathrm{Int}(E, D)$ (Proposition 2.3), every prime above $\mathfrak{m}$ in $\mathrm{Int}(E, D)$ lifts in a prime above $\mathfrak{m}'$ in $\mathrm{Int}(E, D')$ and hence is of the type $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, D) | f(\alpha) \in \hat{\mathfrak{m}}'\}$. Moreover, if $\alpha \neq \beta$ there is $f \in \mathrm{Int}(E, D')$ such that $f(\alpha) \in \hat{\mathfrak{m}}'$ but $f(\beta) \notin \hat{\mathfrak{m}}'$ and there is an integer n such that $f^n \in \mathrm{Int}(E, D)$ (Proposition 2.6), thus $f^n(\alpha) \in \hat{\mathfrak{m}}'$ but $f^n(\beta) \notin \hat{\mathfrak{m}}'$; therefore, $\mathfrak{M}_\alpha \neq \mathfrak{M}_\beta$.

Remarks 3.5. (i) If E is not a fractional subset of D' then $\mathrm{Int}(E, D) = D$ since $\mathrm{Int}(E, D) \subset \mathrm{Int}(E, D')$ and $\mathrm{Int}(E, D') = D'$ [19, Lemma 2.0]. It remains to study the case where E is a fractional subset of D' but not of D.

(ii) Theorem 3.4 does notably improve the first statement of Theorem 3.1 of [19] that says only that if D is unibranched (hence D' is the ring of a valuation v) and α, β are distinct elements of D, then $\mathfrak{M}_\alpha \neq \mathfrak{M}_\beta$.

(iii) If D is analytically irreducible, the primes of $\mathrm{Int}(D)$ above the maximal ideal $\mathfrak{m}$ of D are in one-to-one correspondence with the elements of the completion $\widehat{D}$ of D *for the $\mathfrak{m}$-adic topology* [11]; this special case fits with the general result of Theorem 3.4 thanks to

Proposition 3.6. *Let D be a (one-dimensional noetherian local) unibranched domain of maximal ideal $\mathfrak{m}$ and $\mathfrak{m}'$ the maximal ideal of D'. The following statements are equivalent:*

(i) *the restriction on D of the $\mathfrak{m}'$-adic is the $\mathfrak{m}$-adic topology;*

(ii) *the $\mathfrak{m}$-adic completion $\widehat{D}$ of D is included in the $\mathfrak{m}'$-adic completion $\widehat{D}'$ of D';*

(iii) *D is analytically irreducible.*

Proof. (i) ⇒ (ii) Clear.

(ii) ⇒ (iii) If the completion $\widehat{D}$ of D is included in the completion $\widehat{D}'$ of D' then $\widehat{D}$ is a domain.

(iii) ⇒ (i) If D is analytically irreducible then D' is a finite D-module [3, 21], hence there is an integer n such that $\mathfrak{m}'^n$ is included in the (nonzero) conductor $[D : D']$, hence $\mathfrak{m}'^n$ is an ideal of D included in $\mathfrak{m}$ and, for every k, $\mathfrak{m}'^{nk}$ is included in $\mathfrak{m}^k$. On the other hand, there is an integer m such that $\mathfrak{m}^m$ is included in $\mathfrak{m}'^n$ (since D is one-dimensional noetherian and local) and, for every k, $\mathfrak{m}^{mk}$ is included in $\mathfrak{m}'^{nk}$. □

4. Noetherian property

We generalize to $\mathrm{Int}(E, D)$ the results on the noetherian property of $\mathrm{Int}(D)$ [19]; as in this particular case, an obvious necessary condition for $\mathrm{Int}(E, D)$ to be noetherian is for D itself to be noetherian (since D is the homomorphic image of $\mathrm{Int}(E, D)$ by the evaluation map at some element α of E). We set first the case of a finite subset E.

Lemma 4.1. *Let D be a domain and E a finite subset of K. Then* $\mathrm{Int}(E, D)$ *is not noetherian.*

Proof. If E contains only one point α then clearly $\mathrm{Int}(E, D) = D + (X - \alpha)K[X]$; it shares with $K[X]$ the ideal $I = (X - \alpha)K[X]$ and it is not noetherian since $K = K[X]/I$ is not finitely generated (as a module) over $D = \mathrm{Int}(E, D)/I$ [9, Proposition 1]. If E contains the points $\alpha_1, \dots, \alpha_n$ then $\mathrm{Int}(E, D)$ shares with $K[X]$ the ideal $I = \varphi K[X]$, where $\varphi = \prod_{i=1}^{n}(X - \alpha_i)$, and the same proof could apply (although it is less straightforward to show that $K[X]/I$ is not finitely generated over $\mathrm{Int}(E, D)/I$). We may also show directly that I is not a finitely generated ideal of $\mathrm{Int}(E, D)$: Assume by way of contradiction that I is generated by the polynomials $f_1, \dots, f_r$. Then any polynomial $f \in I$ is such that $f = \sum_{i=1}^{r} h_i f_i$, where $h_i \in \mathrm{Int}(E, D)$, for $1 \le i \le r$. Writing $f = \varphi g$ and $f_i = \varphi g_i$ for $1 \le i \le r$ and dividing by φ, we have $g = \sum_{i=1}^{r} h_i g_i$; therefore, $g(\alpha_1)$ is in the D-module generated by $g_1(\alpha_1), \dots, g_r(\alpha_1)$. We reach a contradiction since g may be any polynomial of $K[X]$. □

The results of the previous lemma allow us also to conclude that for an infinite fractional subset E of a unibranched domain D, some primes of $\mathrm{Int}(E, D)$ are not finitely generated.

Lemma 4.2. *Let D be a (one-dimensional noetherian local) unibranched domain with finite residue field, E an infinite fractional subset of D, $\mathfrak{m}'$ the maximal ideal of D', and $\overline{E}$ the topological closure of E in the completion $\hat{K}$ of K for the topology defined by the valuation of D'. Then*

(i) *there exists a point of accumulation in $\overline{E}$ for this topology;*

(ii) *if α is a point of accumulation of $\overline{E}$ for this topology, the prime $\mathfrak{M}_\alpha = \{f \in \mathrm{Int}(E, D) | f(\alpha) \in \hat{\mathfrak{m}}'\}$ of $\mathrm{Int}(E, D)$ is not finitely generated.*

Proof. (i) Since E is fractional, it is a subset of the compact space $(1/d)\overline{D}$.

(ii) Assume by way of contradiction that $\mathfrak{M}_\alpha$ is generated by the polynomials $f_1, \dots, f_r$. Then $f_i(\alpha) \in \mathfrak{m}'$, for $1 \le i \le r$, and if β is close enough to α in $\overline{E}$, $f_i(\beta) \in \mathfrak{m}'$ for $1 \le i \le r$. Hence the inclusion $\mathfrak{M}_\alpha \subset \mathfrak{M}_\beta$, however, $\mathfrak{M}_\alpha$ is maximal and distinct from $\mathfrak{M}_\beta$ (Theorem 3.4). □

We can now generalize to $\mathrm{Int}(E, D)$ [19, Theorem 2.3].

Theorem 4.3. *Let D be a noetherian domain and E a fractional subset of D. If* $\mathrm{Int}(E, D)$ *is noetherian then there is no height-one prime in D' with finite residue field.*

Proof. Assume by way of contradiction that a maximal ideal $\mathfrak{m}'$ of D' is height-one with finite residue field. We let $\mathfrak{m} = \mathfrak{m}' \cap D$. There are only finitely many primes of D' above $\mathfrak{m}$ (one of them being $\mathfrak{m}'$) [21, (33.10)], and there is

a ring R, finitely generated as a D-module, such that $D \subset R \subset D'$, and $\mathfrak{m}'$ is the only prime of D' above $\mathfrak{n} = \mathfrak{m}' \cap R$ (R is obtained by adjoining to D an element which belongs to $\mathfrak{m}'$ but to no other prime above $\mathfrak{m}$). Any element of the (nonzero) conductor $[D : R]$ is clearly in the conductor $[\mathrm{Int}(E, D) : \mathrm{Int}(E, R)]$, hence if $\mathrm{Int}(E, D)$ is noetherian then $\mathrm{Int}(E, R)$ is a finitely generated $\mathrm{Int}(E, D)$-module and thus a noetherian domain; by localization, $\mathrm{Int}(E, R_{\mathfrak{n}}) = (\mathrm{Int}(E, R))_{\mathfrak{n}}$ (Proposition 1.5) is also noetherian. We may suppose E is infinite according to Lemma 4.1, and we reach a contradiction according to Lemma 4.2 since $R_{\mathfrak{n}}$ is clearly a (one-dimensional noetherian local) unibranched domain with finite residue field.

Remarks 4.4. (i) If E is not a fractional subset of D' then $\mathrm{Int}(E, D) = D$ [20, Lemma 2.0]. As above (Remark 3.5(i)), it remains to study the case where E is a fractional subset of D' but not of D.

(ii) If E is a finite subset of D then it is a fractional subset.

(iii) To finish the proof of Theorem 4.3 for a subset E of D (without using Lemma 4.2), we could alternatively say that the integral closure of $\mathrm{Int}(E, R_{\mathfrak{n}})$ is $\mathrm{Int}(E, D'_{\mathfrak{m}'})$ (Proposition 2.3) and that $\dim \mathrm{Int}(E, R_{\mathfrak{n}}) = \dim \mathrm{Int}(E, D'_{\mathfrak{m}'}) = 2$ (Corollary 1.4). Therefore, if $\mathrm{Int}(E, R_{\mathfrak{n}})$ is noetherian then $\mathrm{Int}(E, D'_{\mathfrak{m}'})$ is a dimension 2 noetherian domain [21, (33.12)]. We would then reach again a contradiction since $\mathrm{Int}(E, D'_{\mathfrak{m}'})$ is also a Prüfer domain, indeed, it does contain the ring of integer-valued polynomials $\mathrm{Int}(D'_{\mathfrak{m}'})$ over a discrete rank-one valuation domain and such a ring is known to be a Prüfer domain [15].

A necessary condition for $\mathrm{Int}(D)$ to be noetherian, as we shall see in Corollary 4.6, is to be included in $D'[X]$. We can characterize this inclusion as follows (and this is a sharper version of [19, Corollary 4.4] where condition (i) is only given as sufficient).

Theorem 4.5. *Let D be a noetherian domain. The following statements are equivalent*:

(i) *every prime of height one in D' has infinite residue field*;
(ii) $\mathrm{Int}(D, D') = D'[X]$;
(iii) $\mathrm{Int}(D) \subset D'[X]$;
(iv) $\mathrm{Int}(D') = D'[X]$.

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{p}'$ be a height-one prime of D' and $\mathfrak{p} = \mathfrak{p}' \cap D$. Then $A/\mathfrak{p}$ is infinite. Indeed, either $\mathfrak{p}$ is not maximal and there is nothing to add or it is maximal, but in this case, $\mathfrak{p}'$ is also maximal in D' and $D'/\mathfrak{p}'$ is a $D/\mathfrak{p}$-module of finite type [21, (33.10)]. Then if $f \in \mathrm{Int}(D, D')$, Cramer's rule shows that $f \in D'_{\mathfrak{p}'}[X]$. Therefore, $f \in D'[X]$ since D' is Krull, and thus $D' = \bigcap\{D'_{\mathfrak{p}'} : \mathfrak{p}' \text{ is height-one}\}$.

(ii) $\Rightarrow$ (iii) Clear from the inclusion $\mathrm{Int}(D) \subset \mathrm{Int}(D, D')$ (Proposition 1.1).

(iii) $\Rightarrow$ (iv) Since D is noetherian, $\mathrm{Int}(D, D')$ is integral over $\mathrm{Int}(D)$ [19, Proposition 2.2], a fortiori its elements are integral over $D'[X]$ that is clearly integrally closed; therefore, $\mathrm{Int}(D') \subset \mathrm{Int}(D, D') \subset D'[X]$. The reverse inclusion is clear (Proposition 1.2).

(iv) $\Rightarrow$ (i) Assume by way of contradiction that a maximal ideal $\mathfrak{m}'$ of D' has height-one with finite residue field. Then $\mathrm{Int}(D') = D'[X]$ implies, by localization, that $\mathrm{Int}(D'_{\mathfrak{m}'}) = D'_{\mathfrak{m}'}[X]$ (Proposition 1.5); however, this last equality does not hold since $D'_{\mathfrak{m}'}$ is a discrete rank-one valuation ring. $\square$

Theorems 4.3 and 4.5 together lead immediately to

Corollary 4.6. *If D is a domain such that* $\mathrm{Int}(D)$ *is noetherian then D is noetherian and* $\mathrm{Int}(D) \subset D'[X]$.

Conversely, if $\mathrm{Int}(D) \subset D'[X]$, D is noetherian, and D' is a finitely generated D-module, then $\mathrm{Int}(D)$ is noetherian.

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