INTEGER-VALUED POLYNOMIALS ON A SUBSET

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Abstract. We let $D$ be a local (noetherian) one-dimensional unibranched domain, $K$ its quotient field, $m$ its maximal ideal, $D'$ its integral closure, and $m'$ the maximal ideal of $D'$. If $E$ is a subset of $K$, we let $\text{Int}(E, D)$ be the set of integer-valued polynomials on $E$, thus $\text{Int}(E, D) = \{f \in K[X] | f(E) \subseteq D\}$. For a fractional subset $E$ of $D$ (i.e., there is a nonzero element $d$ of $D$ such that $dE \subseteq D$), we show that the prime ideals of $\text{Int}(E, D)$ above $m$ are in one-to-one correspondence with the elements of the topological closure of $E$ in the completion of $K$ for the $m'$-adic topology.

Introduction

Throughout this paper, $D$ is a domain; we denote by $K$ its quotient field and by $D'$ the integral closure of $D$. If $E$ is a subset of $K$, we let $\text{Int}(E, D)$ be the set of integer-valued polynomials on $E$, thus $\text{Int}(E, D) = \{f \in K[X] | f(E) \subseteq D\}$; it is clearly a ring containing $D$, throughout, we assume that $E$ is nonempty and $D$ is not a field, hence that $\text{Int}(E, D)$ is strictly contained in $K[X]$ (as indeed $\text{Int}(E, D) \cap K = D$). If $E$ is $D$ itself, we simply write $\text{Int}(D)$ for $\text{Int}(D, D)$.

The very classical case is that of The Ring of integer-valued polynomials, i.e., $\text{Int}(\mathbb{Z})$, or even $\text{Int}(D)$ where $D$ is the ring of integers of a number field [22, 23] and, even more generally, where $D$ is Dedekind [5]; by localization, $D$ turns into a discrete rank-one valuation domain and the prime ideals of $\text{Int}(D)$ above the maximal ideal $m$ of $D$ are then known to be in one-to-one correspondence with the elements of the completion $\hat{D}$ of $D$: to any element $\alpha$ of $\hat{D}$ corresponds the prime $\mathfrak{m}_\alpha = \{f \in \text{Int}(D) | f(\alpha) \in \hat{m}\}$ [12, 4, 13], considering the $m$-adic topology, this result generalizes to the case where $D$ is noetherian, local, one-dimensional, with finite residue field, and is analytically irreducible (that is where $\hat{D}$ is a domain) [11]. Dropping this last hypothesis, however, only two facts were known in the previous decade:

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(1) The prime of $\text{Int}(D)$ above $m$ are of the type $\mathfrak{m}_\alpha = \{ f \in \text{Int}(D) | f(\alpha) \in \widehat{m} \}$ [13].

(2) If $D$ is not analytically irreducible, then some of those primes are equal; namely, if $(\alpha - \beta)$ is a zero divisor in $\widehat{D}$, then $\mathfrak{m}_\alpha = \mathfrak{m}_\beta$ [14].

Letting $D$ be noetherian, one-dimensional, and local, we say that $D$ is unibranched if, moreover, $D'$ is local (hence a discrete rank-one valuation domain); recall that $D$ is analytically irreducible if and only if it is unibranched and $D'$ is a finite $D$-module [3, 21]. Very recently it has been shown that if $D$ is not unibranched (first in the case where $D'$ is a finite $D$-module [9] and then without this hypothesis [19]), there are only finitely many primes above $m$ (namely, $\mathfrak{m}_\alpha = \mathfrak{m}_\beta$ if $(\alpha - \beta)$ lies in some nontrivial ideal of $D$); hence the unibranched (and nonanalytically irreducible) case was the only one to remain open. So, letting $D$ be a unibranched domain (with finite residue field), we determine here entirely the spectrum of $\text{Int}(E, D)$ for any fractional subset $E$ of $D$ (i.e., such that $\mathfrak{m}E \subset D$ for a nonzero element $d$ of $D$), recovering the spectrum of $\text{Int}(D)$ (i.e., $E = D$) as well as the classical results on a discrete rank-one valuation domain (i.e., $D = D'$) as special cases.

In the first section we state some generalities on integer-valued polynomials on a subset, mostly on Krull dimension and localization. In §2, generalizing a result of Gilmer, Heinzer, and Lantz [19], we show that if $D$ is noetherian, one-dimensional, local, with finite residue field, then, for any fractional subset $E$ of $D$, $\text{Int}(E, D')$ is the integral closure of $\text{Int}(E, D)$ and that if, moreover, $D$ is unibranched then $\text{Int}(E, D')$ is a radical extension of $\text{Int}(E, D)$ (i.e., every element of $\text{Int}(E, D')$ has a power in $\text{Int}(E, D)$). In §3 we first determine the spectrum of $\text{Int}(E, D)$ for a fractional subset $E$ of a discrete rank-one valuation domain and then of a unibranched domain $D$ (whose integral closure $D'$ is a discrete rank-one valuation domain) as follows: Letting $\mathfrak{m}'$ be the maximal ideal of $D'$, we consider the $\mathfrak{m}'$-adic topology, hence the topology defined by the valuation of $D'$, rather than the $m$-adic topology, treating $D$ as a subset of $D'$; we can thus deduce the spectrum of $\text{Int}(D)$ from that of $\text{Int}(D, D')$ and more generally establish that for any fractional subset $E$ of $D$ the primes of $\text{Int}(E, D)$ above $m$ are in one-to-one correspondence with the elements of the topological closure of $E$ in the completion of $K$ for the topology defined by the valuation of $D'$. Lastly we discuss the noetherian property of $\text{Int}(E, D)$, generalizing and shedding some light on the results of [19] obtained in the particular case of $\text{Int}(D)$.

1. Krull dimension and localization

We start with some elementary inclusions.

**Proposition 1.1.** If $D$ and $B$ are two domains of quotient field $K$ such that $D \subset B$ and $E$, $F$ two subsets of $K$ and such that $E \subset F$, then $\text{Int}(F, D) \subset \text{Int}(E, B)$.

If $E$ is not contained in $D$ then the polynomial $X$ is not in $\text{Int}(E, D)$, thus:

**Proposition 1.2.** Let $D$ be a domain and $E$ a subset of $K$. Then the following statements are equivalent:

(i) $E \subset D$. 
(ii) \( \text{Int}(D) \subset \text{Int}(E, D) \).
(iii) \( D[X] \subset \text{Int}(E, D) \).

It is easy to describe some primes of \( \text{Int}(E, D) \): If \( \alpha \in E \) and \( p \) is a prime ideal of \( D \), we note \( \mathfrak{P}(\alpha, p) = \{ f \in \text{Int}(E, D) | f(\alpha) \in p \} \) the set of polynomials of \( \text{Int}(E, D) \) taking on the value of \( \alpha \) in \( p \); this set is clearly a prime ideal of \( \text{Int}(E, D) \) above \( p \) and the quotient \( \text{Int}(E, D)/\mathfrak{P}(\alpha, p) \) is isomorphic to \( D/p \); in particular, if \( m \) is a maximal ideal of \( D \) then \( \mathfrak{M}_\alpha = \mathfrak{P}(\alpha, m) \) is a maximal ideal of \( \text{Int}(E, D) \). If \( p \subseteq q \) then \( \mathfrak{P}(\alpha, p) \subseteq \mathfrak{P}(\alpha, q) \), and we can generalize here to \( \text{Int}(E, D) \) the results known for the Krull dimension of \( \text{Int}(D) \) [5, 13] (we denote by \( \dim R \) the Krull dimension of a ring \( R \)).

First note that if \( E \) is too large it may happen that every polynomial in \( \text{Int}(E, D) \) is constant (i.e., \( \text{Int}(E, D) = D \)). Also, following McQuillan [20], we shall say that \( E \) is a fractional subset of \( D \) if there exists a nonzero element \( d \) of \( D \) such that \( dE \subset D \).

**Proposition 1.3.** Let \( D \) be a domain and \( E \) a subset of \( K \). Then \( \dim \text{Int}(E, D) \geq \dim D \); if, moreover, \( E \) is a fractional subset of \( D \) then \( \dim \text{Int}(E, D) \geq \dim D + 1 \).

**Proof.** Let \( (0) = p_0 \subset p_1 \subset \cdots \subset p_d \) be a chain of primes of \( D \); it gives rise to the chain of primes \( \mathfrak{P}(\alpha, p_0) \subset \mathfrak{P}(\alpha, p_1) \subset \cdots \subset \mathfrak{P}(\alpha, p_d) \) of \( \text{Int}(E, D) \). Moreover, if \( d \) is such that \( dE \subset D \) then \( d(X - \alpha) \in \mathfrak{P}(\alpha, p_0) \), hence \( (0) \subseteq \mathfrak{P}(\alpha, p_0) \). \( \square \)

If \( R \) is a domain, recall that its valuative dimension \( \dim_v R \) is the supremum of the Krull dimensions of the overrings of \( R \) in the quotient field of \( R \); Also recall that a **Jaffard domain** is a domain such that \( \dim_v R = \dim R \); and lastly recall that a noetherian domain is a Jaffard domain [1].

**Corollary 1.4.** Let \( D \) be a domain and \( E \) a subset of \( K \). Then

(i) \( \dim \text{Int}(E, D) \leq \dim_v D + 1 \);
(ii) if \( D \) is a Jaffard domain then \( \dim \text{Int}(E, D) \leq \dim D + 1 \);
(iii) if \( D \) is a Jaffard domain and \( E \) a fractional subset of \( D \), then \( \dim \text{Int}(E, D) = \dim D + 1 \).

**Proof.** (i) \( \text{Int}(E, D) \) satisfies the inclusions \( D \subset \text{Int}(E, D) \subset K[X] \), and the result follows from [2, Lemma 1.1]; (ii) and (iii) are clear. \( \square \)

We conclude this section with localization properties:

**Proposition 1.5.** Let \( D \) be a domain, \( E \) a subset of \( K \), and \( S \) a multiplicative subset of \( D \). Then

(i) \( S^{-1} \text{Int}(E, D) \subset \text{Int}(E, S^{-1}D) \);
(ii) if \( E \) is a subset of \( D \) and \( D \) is noetherian, then \( S^{-1} \text{Int}(E, D) = \text{Int}(E, S^{-1}D) \);
(iii) if \( A = E \) is a subring of \( D \) and \( S \) is a multiplicative subset of \( A \), then \( S^{-1} \text{Int}(A, D) \subset \text{Int}(S^{-1}A, S^{-1}D) \). Moreover, if \( A \) or \( D \) is noetherian, then \( S^{-1} \text{Int}(A, D) = \text{Int}(S^{-1}A, S^{-1}D) = \text{Int}(A, S^{-1}D) \).

**Proof.** (i) Immediate.
(ii) Let \( f \in \text{Int}(E, S^{-1}D) \). The \( D \)-module generated by \( f(E) \) is (included in) a finite \( D \)-module (generated by the coefficients of \( f \)); hence there exists \( s \in S \) such that \( sf(E) \subset S^{-1}D \), thus \( f \in S^{-1}\text{Int}(E, D) \).

(iii) If \( f \in \text{Int}(A, D) \), \( f \) is by definition a polynomial with coefficients in \( K[X] \) whose homomorphic image in \( (K/D)[X] \) is null on \( A \), then its image in \( S^{-1}(K/D)[X] \) is null on \( S^{-1}A \) [10, Proposition 4], thus \( \text{Int}(A, D) \subset \text{Int}(S^{-1}A, S^{-1}D) \) and the following inclusions always hold: \( S^{-1}\text{Int}(A, D) \subset \text{Int}(S^{-1}A, S^{-1}D) \subset \text{Int}(A, S^{-1}D) \). Moreover, if \( D \) is noetherian it follows from (ii) that \( \text{Int}(A, S^{-1}D) \subset S^{-1}\text{Int}(A, D) \); the same holds if \( A \) is noetherian, with the same proof, considering \( A \)-modules instead of \( D \)-modules. □

Remark 1.6. If \( E \) is simply a subset of \( K \) (and not of \( D \)), then \( f(E) \) is not necessarily included in the \( D \)-module generated by the coefficients of \( f \) and the conclusion of (ii) may fail. Indeed if, for instance, \( D = \mathbb{Z} \), \( S \) is any nontrivial multiplicative set, and \( E = S^{-1}\mathbb{Z} \), then \( \text{Int}(E, S^{-1}\mathbb{Z}) = S^{-1}\text{Int}(\mathbb{Z}) \) does contain nonconstant polynomials.

2. INTEGRAL CLOSURE

We start with a general condition for \( \text{Int}(E, D) \) to be integrally closed.

**Proposition 2.1.** Let \( D \) be a domain and \( E \) a subset of \( K \). Then \( \text{Int}(E, D) \) is integrally closed if and only if \( D \) is integrally closed.

**Proof.** • If \( D \) is not integrally closed there exists \( x \in K, x \notin D \), which is integral over \( D \), then \( x \notin \text{Int}(E, D) \) but \( x \) is integral over \( \text{Int}(E, D) \).

• If \( D \) is integrally closed, and if \( f \in K(X) \) is integral over \( \text{Int}(E, D) \), then \( f \in K(X) \) (since \( f \) is integral over \( K[X] \), which is integrally closed), and, for every \( x \in E \), \( f(x) \) is integral over \( D \); hence \( f(x) \in D \), thus \( f \in \text{Int}(E, D) \). □

In [19] it is proved that if \( D \) is a one-dimensional noetherian domain then the elements of \( \text{Int}(D') \) are integral over \( \text{Int}(D) \); this is done by assuming (without loss of generality) that \( D \) is local. Here we state a similar result for \( \text{Int}(E, D) \), but first we present the following

**Lemma 2.2.** Let \( D \) be a noetherian domain, \( E \) a fractional subset of \( D \), and \( f \in \text{Int}(E, D') \). Then \( f(E) \) is contained in a \( D \)-algebra \( R \), finitely generated as a \( D \)-module and such that \( D \subset R \subset D' \).

**Proof.** Let \( d \) be a nonzero element such that \( dE \subset D \), and let \( f = a_0/b + (a_1/b)X + \cdots + (a_n/b)X^n \), where \( a_0, a_1, \ldots, a_n \) and \( b \) are in \( D \). If \( x \in E \) it is clear that \( bd^n f(x) \in D \), hence \( f(E) \) is included in the \( D \)-module \( M = D' \cap (1/bd^n)D \). Since \( D \) is noetherian, \( M \) has finitely many generators \( x_1, \ldots, x_r \) and \( f(E) \) is included in the \( D \)-algebra \( R = D[x_1, \ldots, x_r] \), which is also a finitely generated \( D \)-module. □

**Proposition 2.3.** Let \( D \) be a one-dimensional noetherian local domain with finite residue field, and let \( E \) be a fractional subset of \( D \). Then \( \text{Int}(E, D') \) is the integral closure of \( \text{Int}(E, D) \).

**Proof.** Since \( \text{Int}(E, D') \) is integrally closed, it remains to prove that it is an integral extension of \( \text{Int}(E, D) \): let \( f \in \text{Int}(E, D') \), \( f(E) \) is contained in a
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Let $D$ be a noetherian domain and $E$ a fractional subset of $D$. Then $\text{Int}(E, D')$ is almost integral over $\text{Int}(E, D)$. 

Proof. Let $f \in \text{Int}(E, D')$, $f(E)$ is contained in a $D$-algebra $R$, finitely generated as a $D$-module such that $D \subset R \subset D'$. Then, for every integer $n$, $f^n(E) \subset R$, and hence, for $d$ in the (nonzero) conductor $[D : R]$, $df^n \in \text{Int}(E, D)$. \(\square\)

Proposition 2.3 can be improved in the case where $D$ is unibranched (generalizing a result implicitly contained in the proof of Theorem 3.1 of [19], with the same argument):

Proposition 2.6. Let $D$ be a (one-dimensional noetherian local) unibranched domain with finite residue field, and let $E$ be a fractional subset of $D$. Then $\text{Int}(E, D')$ is a radical extension of $\text{Int}(E, D)$: for any $f \in \text{Int}(E, D')$ there is an integer $n$ such that $f^n \in \text{Int}(E, D)$. 

Proof. As above, let $f \in \text{Int}(E, D')$, $f(E)$ is contained in a $D$-algebra $R$, finitely generated as a $D$-module such that $D \subset R \subset D'$. Since in this case $D$ is unibranched then $R$ is local, say with maximal ideal $n$ and some power $n^n$ of $n$ is contained in the (nonzero) conductor $[D : R]$, if $q$ is the cardinal of $R/n$ then, for every $x \in R$, $x^{q-1} = \varepsilon + y$ (where $\varepsilon$ is 0 or 1 and $y \in n$); if $p$ is the characteristic of $R/n$, it follows from a basic property of binomial coefficients that $(x^{q-1})^p = \varepsilon + y$, where $\varepsilon$ is 0 or 1 and $y \in n^2$; by induction, letting $n = (q - 1)p^{m-1}$, we have $x^n = \varepsilon + y$, where $y \in n^m$, hence $x^n \in D$, thus $f^n \in \text{Int}(E, D)$. \(\square\)

3. INTEGER-VALUED POLYNOMIALS OVER A UNIBRANCHED DOMAIN

If $D$ is a local domain of maximal ideal $m$ with finite residue field and $E$ is a subset of its quotient field $K$, we let $\mathcal{J} = \{f \in \text{Int}(E, D)| f(E) \subset m\}$; $\mathcal{J}$ is clearly an ideal of $\text{Int}(E, D)$. We first establish the following (generalizing [9]):

Lemma 3.1. Let $D$ be a local domain with maximal ideal $m$ and finite residue field, $E$ a subset of $K$, and $\mathfrak{M}$ a prime ideal of $\text{Int}(E, D)$ containing the ideal...
\( \mathcal{J} = \{ f \in \text{Int}(E, D) | f(E) \subseteq m \} \). Then \( \mathcal{M} \) is maximal, it is above \( m \), and its residue field is isomorphic to \( D/m \).

**Proof.** \( \mathcal{M} \) is clearly above \( m \), and if \( q \) denotes the cardinal of \( D/m \) then, for any \( f \) in \( \text{Int}(E, D) \), \( (f^q - f) \in \mathcal{J} \); hence the residue field of \( \mathcal{M} \) is of cardinal \( q \). Therefore, it is isomorphic to \( D/m \) and \( \mathcal{M} \) is maximal. \( \square \)

Now if \( V \) is a discrete rank-one valuation domain with finite residue field we generalize to \( \text{Int}(E, V) \) the classical results on the spectrum of \( \text{Int}(V) \) [13]. In a first step we suppose \( E \) to be a subset contained in \( V \):

**Lemma 3.2.** Let \( E \) be a subset of a rank-one discrete valuation domain \( V \) with maximal ideal \( n \) and with finite residue field. Then the primes of \( \text{Int}(E, V) \) above \( n \) are in one-to-one correspondence with the elements of the topological closure \( \overline{E} \) of \( E \) in \( \hat{V} \): to any element \( \alpha \) of \( \overline{E} \) corresponds the prime \( \mathfrak{m}_\alpha = \{ f \in \text{Int}(E, V) | f(\alpha) \in n \} \).

**Proof.** This is an easy repetition of the classical proof [13]. \( n \) is a principal ideal generated by an element \( \pi \), hence \( \mathcal{J} = \{ f \in \text{Int}(E, V) | f(E) \subseteq n \} = \pi \text{Int}(E, V) \). Now every integer-valued polynomial (on \( E \)) is a continuous function on the topological closure \( \overline{E} \) of \( E \) with values in the completion \( \hat{V} \) of \( V \), thus

\[
\text{Int}(V) \subseteq \text{Int}(E, V) \subseteq \mathcal{C}(\overline{E}, \hat{V}).
\]

The ideal \( \mathcal{J} \) is the intersection \( \text{Int}(E, V) \cap \mathcal{C}(\overline{E}, \hat{n}) \), thus

\[
\text{Int}(E, V)/\mathcal{J} \subseteq \mathcal{C}(\overline{E}, \hat{V})/\mathcal{C}(\overline{E}, \hat{n}) = \mathcal{C}(\overline{E}, k)
\]

(where \( \mathcal{C}(\overline{E}, k) \) is the ring of locally constant functions on \( \overline{E} \) with values in \( \hat{V}/\hat{n} = k \)). Since \( \overline{E} \) is compact (being a subspace of \( \hat{V} \)), the primes of \( \mathcal{C}(\overline{E}, k) \) are in one-to-one correspondence with the elements of \( \overline{E} \): to any element \( x \) corresponds the set of functions null at \( x \) [3, II, §4, exercise 17].

Every prime of \( \text{Int}(E, V) \) above \( n \) contains \( \pi \), hence it contains \( \mathcal{J} \) and, therefore, it is maximal (Lemma 2.1); hence every prime of \( \text{Int}(E, V)/\mathcal{J} \) lifts in \( \mathcal{C}(\overline{E}, k) \) (therefore, every prime of \( \text{Int}(E, V) \) lifts in \( \mathcal{C}(\overline{E}, \hat{V}) \)). In conclusion the primes of \( \text{Int}(E, V) \) containing \( n \) are of the type

\[
\mathfrak{m}_\alpha = \{ f \in \text{Int}(E, V) | f(\alpha) \in \hat{n} \}, \quad \text{where} \quad \alpha \in \overline{E}.
\]

Lastly, if \( \alpha \neq \beta \), there is \( f \) in \( \text{Int}(V) \) such that \( f(\alpha) \in \hat{n} \) but \( f(\beta) \notin \hat{n} \) (because \( \text{Int}(V) \) is dense in \( \mathcal{C}(\hat{V}, \hat{V}) \) [11, 13]), hence \( \mathfrak{m}_\alpha \neq \mathfrak{m}_\beta \), since \( \text{Int}(V) \subseteq \text{Int}(E, V) \). \( \square \)

The previous lemma generalizes to a fractional subset:

**Proposition 3.3.** Let \( E \) be a fractional subset of a rank-one discrete valuation domain \( V \) with maximal ideal \( n \) and finite residue field. Then the primes of \( \text{Int}(E, V) \) above \( n \) are in one-to-one correspondence with the elements of the topological closure \( \overline{E} \) of \( E \) in the completion \( \hat{K} \) of the quotient field \( K \) of \( V \): to any element \( \alpha \) of \( \overline{E} \) corresponds the prime \( \mathfrak{m}_\alpha = \{ f \in \text{Int}(E, V) | f(\alpha) \in \hat{n} \} \).

**Proof.** Since \( E \) is a fractional subset of \( V \), there exists an element \( d \) of \( V \) such that \( dE \subseteq V \); we let \( F = dE \) and \( \phi \) be the ring homomorphism such that \( \phi(f(X)) = f(X/d) \), \( \phi \) is clearly an isomorphism from \( \text{Int}(E, V) \) onto
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Int(F, V). Now F is a subset of V and from Lemma 3.2 the primes of Int(F, V) above n are in one-to-one correspondence with the elements of the topological closure \( \overline{F} \) of \( F \) in the completion \( \widehat{V} \) of \( V \) (contained in the completion \( \widehat{K} \) of \( K \)). If \( E \) is the topological closure of \( E \) in \( \widehat{K} \) then \( \overline{F} = dE \).

Let \( \alpha \) be an element of \( \overline{F} \), \( \beta = \alpha/d \), and \( \mathfrak{n}_\alpha = \{ f \in \text{Int}(F, V) | f(\alpha) \in \mathfrak{n} \} \). Then \( \phi^{-1}(\mathfrak{n}_\alpha) = \mathfrak{m}_\beta = \{ f \in \text{Int}(E, V) | f(\beta) \in \mathfrak{n} \} \).

We can generalize further to a unibranched domain:

**Theorem 3.4.** Let \( D \) be a (one-dimensional noetherian local) unibranched domain with maximal ideal \( \mathfrak{m} \) and finite residue field. Let \( \mathfrak{m}' \) be the maximal ideal of \( D' \) and \( E \) be a fractional subset of \( D \). The primes of Int(\( E \), \( D \)) above \( \mathfrak{m} \) are in one-to-one correspondence with the elements of the topological closure \( \overline{E} \) of \( E \) in the completion \( \widehat{K} \) of \( K \) for the topology defined by the valuation of \( D' \): to any element \( \alpha \) of \( \overline{E} \) corresponds the prime \( \mathfrak{m}_\alpha = \{ f \in \text{Int}(E, D) | f(\alpha) \in \mathfrak{m}' \} \).

**Proof.** We consider the inclusion Int(\( E \), \( D \)) \( \subset \) Int(\( E \), \( D' \) ). Since Int(\( E \), \( D' \)) is integral over Int(\( E \), \( D \)) (Proposition 2.3), every prime above \( \mathfrak{m} \) in Int(\( E \), \( D \)) lifts in a prime above \( \mathfrak{m}' \) in Int(\( E \), \( D' \)) and hence is of the type \( \mathfrak{m}_\alpha = \{ f \in \text{Int}(E, D) | f(\alpha) \in \mathfrak{m}' \} \). Moreover, if \( \alpha \neq \beta \) there is \( f \in \text{Int}(E, D') \) such that \( f(\alpha) \in \mathfrak{m}' \) but \( f(\beta) \notin \mathfrak{m}' \) and there is an integer \( n \) such that \( f^n \in \text{Int}(E, D) \) (Proposition 2.6), thus \( f^n(\alpha) \in \mathfrak{m}' \) but \( f^n(\beta) \notin \mathfrak{m}' \); therefore, \( \mathfrak{m}_\alpha \neq \mathfrak{m}_\beta \).

**Remarks 3.5.** (i) If \( E \) is not a fractional subset of \( D' \) then Int(\( E \), \( D \)) = \( D \) since Int(\( E \), \( D \)) \( \subset \) Int(\( E \), \( D' \) ) and Int(\( E \), \( D' \)) = \( D' \) [19, Lemma 2.0]. It remains to study the case where \( E \) is a fractional subset of \( D' \) but not of \( D \).

(ii) Theorem 3.4 does notably improve the first statement of Theorem 3.1 of [19] that says only that if \( D \) is unibranched (hence \( D' \) is the ring of a valuation \( v \)) and \( \alpha, \beta \) are distinct elements of \( D \), then \( \mathfrak{m}_\alpha \neq \mathfrak{m}_\beta \).

(iii) If \( D \) is analytically irreducible, the primes of Int(\( D \)) above the maximal ideal \( \mathfrak{m} \) of \( D \) are in one-to-one correspondence with the elements of the completion \( \widehat{D} \) of \( D \) for the \( \mathfrak{m} \)-adic topology [11]; this special case fits with the general result of Theorem 3.4 thanks to

**Proposition 3.6.** Let \( D \) be a (one-dimensional noetherian local) unibranched domain of maximal ideal \( \mathfrak{m} \) and \( \mathfrak{m}' \) the maximal ideal of \( D' \). The following statements are equivalent:

(i) the restriction on \( D \) of the \( \mathfrak{m}' \)-adic is the \( \mathfrak{m} \)-adic topology;

(ii) the \( \mathfrak{m} \)-adic completion \( \widehat{D} \) of \( D \) is included in the \( \mathfrak{m}' \)-adic completion \( \widehat{D}' \) of \( D' \);

(iii) \( D \) is analytically irreducible.

**Proof.** (i) \( \Rightarrow \) (ii) Clear.

(ii) \( \Rightarrow \) (iii) If the completion \( \widehat{D} \) of \( D \) is included in the completion \( \widehat{D}' \) of \( D' \) then \( \widehat{D} \) is a domain.

(iii) \( \Rightarrow \) (i) If \( D \) is analytically irreducible then \( D' \) is a finite \( D \)-module [3, 21], hence there is an integer \( n \) such that \( \mathfrak{m}^n \) is included in the (nonzero) conductor \( [D : D'] \), hence \( \mathfrak{m}^n \) is an ideal of \( D \) included in \( \mathfrak{m} \) and, for every \( k \), \( \mathfrak{m}^{nk} \) is included in \( \mathfrak{m}^k \). On the other hand, there is an integer \( m \) such that \( \mathfrak{m}^m \) is included in \( \mathfrak{m}'' \) (since \( D \) is one-dimensional noetherian and local) and, for every \( k \), \( \mathfrak{m}^{mk} \) is included in \( \mathfrak{m}'^{mk} \). □
4. Noetherian property

We generalize to \( \text{Int}(E, D) \) the results on the noetherian property of \( \text{Int}(D) \) [19]; as in this particular case, an obvious necessary condition for \( \text{Int}(E, D) \) to be noetherian is for \( D \) itself to be noetherian (since \( D \) is the homomorphic image of \( \text{Int}(E, D) \) by the evaluation map at some element \( \alpha \) of \( E \)). We set first the case of a finite subset \( E \).

**Lemma 4.1.** Let \( D \) be a domain and \( E \) a finite subset of \( K \). Then \( \text{Int}(E, D) \) is not noetherian.

*Proof.* If \( E \) contains only one point \( \alpha \) then clearly \( \text{Int}(E, D) = D + (X - \alpha)K[X] \); it shares with \( K[X] \) the ideal \( I = (X - \alpha)K[X] \) and it is not noetherian since \( K = K[X]/I \) is not finitely generated (as a module) over \( D = \text{Int}(E, D)/I \) [9, Proposition 1]. If \( E \) contains the points \( \alpha_1, \ldots, \alpha_n \) then \( \text{Int}(E, D) \) shares with \( K[X] \) the ideal \( I = \varphi K[X] \), where \( \varphi = \prod_{i=1}^{n}(X - \alpha_i) \), and the same proof could apply (although it is less straightforward to show that \( K[X]/I \) is not finitely generated over \( \text{Int}(E, D)/I \)). We may also show directly that \( I \) is not a finitely generated ideal of \( \text{Int}(E, D) \): Assume by way of contradiction that \( I \) is generated by the polynomials \( f_1, \ldots, f_r \). Then any polynomial \( f \in I \) is such that \( f = \sum_{i=1}^{r} h_i f_i \), where \( h_i \in \text{Int}(E, D) \), for \( 1 \leq i \leq r \). Writing \( f = \varphi g \) and \( f_i = \varphi g_i \) for \( 1 \leq i \leq r \) and dividing by \( \varphi \), we have \( g = \sum_{i=1}^{r} h_i g_i \); therefore, \( g(\alpha_1) \) is in the \( D \)-module generated by \( g_1(\alpha_1), \ldots, g_r(\alpha_1) \). We reach a contradiction since \( g \) may be any polynomial of \( K[X] \). \( \square \)

The results of the previous lemma allow us also to conclude that for an infinite fractional subset \( E \) of a unibranched domain \( D \), some primes of \( \text{Int}(E, D) \) are not finitely generated.

**Lemma 4.2.** Let \( D \) be a (one-dimensional noetherian local) unibranched domain with finite residue field, \( E \) an infinite fractional subset of \( D \), \( m' \) the maximal ideal of \( D' \), and \( \bar{E} \) the topological closure of \( E \) in the completion \( K \) of \( K \) for the topology defined by the valuation of \( D' \). Then

(i) there exists a point of accumulation in \( \bar{E} \) for this topology;

(ii) if \( \alpha \) is a point of accumulation of \( \bar{E} \) for this topology, the prime \( \mathfrak{m}_\alpha = \{ f \in \text{Int}(E, D) | f(\alpha) \in m' \} \) of \( \text{Int}(E, D) \) is not finitely generated.

*Proof.* (i) Since \( E \) is fractional, it is a subset of the compact space \( (1/d)\bar{D} \).

(ii) Assume by way of contradiction that \( \mathfrak{m}_\alpha \) is generated by the polynomials \( f_1, \ldots, f_r \). Then \( f_i(\alpha) \in m' \), for \( 1 \leq i \leq r \); and if \( \beta \) is close enough to \( \alpha \) in \( \bar{E} \), \( f_i(\beta) \in m' \) for \( 1 \leq i \leq r \). Hence the inclusion \( \mathfrak{m}_\alpha \subset \mathfrak{m}_\beta \), however, \( \mathfrak{m}_\alpha \) is maximal and distinct from \( \mathfrak{m}_\beta \) (Theorem 3.4). \( \square \)

We can now generalize to \( \text{Int}(E, D) \) [19, Theorem 2.3].

**Theorem 4.3.** Let \( D \) be a noetherian domain and \( E \) a fractional subset of \( D \). If \( \text{Int}(E, D) \) is noetherian then there is no height-one prime in \( D' \) with finite residue field.

*Proof.* Assume by way of contradiction that a maximal ideal \( m' \) of \( D' \) is height-one with finite residue field. We let \( m = m' \cap D \). There are only finitely many primes of \( D' \) above \( m \) (one of them being \( m' \)) [21, (33.10)], and there is
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a ring $R$, finitely generated as a $D$-module, such that $D \subset R \subset D'$, and $m'$ is the only prime of $D'$ above $n = m' \cap R$ ($R$ is obtained by adjoining to $D$ an element which belongs to $m'$ but to no other prime above $m$). Any element of the (nonzero) conductor $[D : R]$ is clearly in the conductor $[\text{Int}(E, D) : \text{Int}(E, R)]$, hence if $\text{Int}(E, D)$ is noetherian then $\text{Int}(E, R)$ is a finitely generated $\text{Int}(E, D)$-module and thus a noetherian domain; by localization, $\text{Int}(E, R_n) = (\text{Int}(E, R))_n$ (Proposition 1.5) is also noetherian. We may suppose $E$ is infinite according to Lemma 4.1, and we reach a contradiction according to Lemma 4.2 since $R_n$ is clearly a (one-dimensional noetherian local) unibranch domain with finite residue field.

Remarks 4.4. (i) If $E$ is not a fractional subset of $D'$ then $\text{Int}(E, D) = D$ [20, Lemma 2.0]. As above (Remark 3.5(i)), it remains to study the case where $E$ is a fractional subset of $D'$ but not of $D$.

(ii) If $E$ is a finite subset of $D$ then it is a fractional subset.

(iii) To finish the proof of Theorem 4.3 for a subset $E$ of $D$ (without using Lemma 4.2), we could alternatively say that the integral closure of $\text{Int}(E, R_n)$ is $\text{Int}(E, D_{m'})$ (Proposition 2.3) and that $\dim \text{Int}(E, R_n) = \dim \text{Int}(E, D_{m'}) = 2$ (Corollary 1.4). Therefore, if $\text{Int}(E, R_n)$ is noetherian then $\text{Int}(E, D_{m'})$ is a dimension 2 noetherian domain [21, (33.12)]. We would then reach again a contradiction since $\text{Int}(E, D_{m'})$ is also a Prüfer domain, indeed, it does contain the ring of integer-valued polynomials $\text{Int}(D_{m'})$ over a discrete rank-one valuation domain and such a ring is known to be a Prüfer domain [15].

A necessary condition for $\text{Int}(D)$ to be noetherian, as we shall see in Corollary 4.6, is to be included in $D'[X]$. We can characterize this inclusion as follows (and this is a sharper version of [19, Corollary 4.4] where condition (i) is only given as sufficient).

Theorem 4.5. Let $D$ be a noetherian domain. The following statements are equivalent:

(i) every prime of height one in $D'$ has infinite residue field;
(ii) $\text{Int}(D, D') = D'[X]$;
(iii) $\text{Int}(D) \subset D'[X]$;
(iv) $\text{Int}(D') = D'[X]$.

Proof. (i) $\Rightarrow$ (ii) Let $p'$ be a height-one prime of $D'$ and $p = p' \cap D$. Then $A/p$ is infinite. Indeed, either $p$ is not maximal and there is nothing to add or it is maximal, but in this case, $p'$ is also maximal in $D'$ and $D'/p'$ is a $D/p$-module of finite type [21, (33.10)]. Then if $f \in \text{Int}(D, D')$, Cramer's rule shows that $f \in D'_{p'}[X]$. Therefore, $f \in D'[X]$ since $D'$ is Krull, and thus $D' = \bigcap \{D'_{p'} : p' \text{ is height-one}\}$.

(ii) $\Rightarrow$ (iii) Clear from the inclusion $\text{Int}(D) \subset \text{Int}(D, D')$ (Proposition 1.1).

(iii) $\Rightarrow$ (iv) Since $D$ is noetherian, $\text{Int}(D, D')$ is integral over $\text{Int}(D)$ [19, Proposition 2.2], a fortiori its elements are integral over $D'[X]$ that is clearly integrally closed; therefore, $\text{Int}(D') \subset \text{Int}(D, D') \subset D'[X]$. The reverse inclusion is clear (Proposition 1.2).

(iv) $\Rightarrow$ (i) Assume by way of contradiction that a maximal ideal $m'$ of $D'$ has height-one with finite residue field. Then $\text{Int}(D') = D'[X]$ implies, by localization, that $\text{Int}(D'_{m'}) = D'_{m'}[X]$ (Proposition 1.5); however, this last equality does not hold since $D'_{m'}$ is a discrete rank-one valuation ring. \(\square\)
Theorems 4.3 and 4.5 together lead immediately to

**Corollary 4.6.** If \( D \) is a domain such that \( \text{Int}(D) \) is noetherian then \( D \) is noetherian and \( \text{Int}(D) \subset D'[X] \).

Conversely, if \( \text{Int}(D) \subset D'[X] \), \( D \) is noetherian, and \( D' \) is a finitely generated \( D \)-module, then \( \text{Int}(D) \) is noetherian.

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**References**


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