UPPER BOUND OF $\sum 1/(a_i \log a_i)$ FOR PRIMITIVE SEQUENCES

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Abstract. A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of $A$ divides any other. The senior author conjectures that, for any primitive sequence $A$,

$$\sum_{a \leq n, a \in A} \frac{1}{a \log a} \leq \sum_{p \leq n} \frac{1}{p \log p} \quad \text{for } n > 1,$$

where $p$ is a variable prime. In our two previous papers we partially proved this conjecture. The main result of this paper is: for any primitive sequence $A$,

$$\sum_{a \in A} \frac{1}{a \log a} < 1.84.$$

We also give a necessary and sufficient condition for this conjecture, i.e.,

$$\sum_{b \in B} \frac{1}{b \log b} \leq \sum_{p \leq n} \frac{1}{p \log p}$$

for any primitive sequence $B$.

1. Introduction

A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of $A$ divides another [3, 4]. We denote by $p_m$ the $m$th prime and by $p$ a variable prime. We define the degree of an integer $a$, denoted by $\Omega(a)$, to be the number of prime factors of $a$, counted with multiplicity. The degree of a primitive sequence $A$, denoted by $d^o(A)$, is defined as the maximum degree of its terms. We take $d^o(A) = 0$ if $A = \{1\}$ or $\emptyset$.

We define $f(A) = \sum_{a \in A} 1/(a \log a)$. We take $f(A) = 0$ if $d^o(A) = 0$. Erdős [1] (cf. [4]) proved that there exists an absolute constant $C$ such that $f(A) < C$ for any primitive sequence, using the inequality

$$\sum_{a \in A} \frac{1}{a} \prod_{p \leq P(a)} \left(1 - \frac{1}{p}\right) \leq 1,$$

where $P(a)$ is the largest prime factor of $a$. It is significant to get a reasonable upper bound for $f(A)$. Using (1) we first got $f(A) < 2.886$; then Robin...
improved it to 2.77 (neither is published). It seems to us that we could not do better by only using (1).

Erdös [2] has asked if the inequality

\[ \sum_{a \leq n, a \in A} \frac{1}{a \log a} \leq \sum_{p \leq n} \frac{1}{p \log p} \quad \text{for } n > 1 \]

is always true for any \( A \). Zhang [7] proved that if \( A \) is primitive with \( d^0(a) \leq 4 \) then (2) is true. For a given primitive sequence \( A \) and \( m \geq 1 \), let \( A_m = \{ a : a \in A, \text{ all prime factors of } a \text{ are } \geq p_m \} \) and \( A'_m = \{ a : a \in A_m, p_m \mid a \} \). Zhang [8] proved that (2) is true for any primitive sequence \( A \) with the property that each \( A_m \) is “homogeneous.” That is, for each \( m \), there is some integer \( s_m \) such that either \( A_m = \emptyset \) or \( \Omega(a) = s_m \) for all \( a \in A_m \).

In this paper we prove the following two theorems.

**Theorem 1.** \( f(A) < 1.84 \) for any primitive sequence \( A = \{ a_i \} \).

**Theorem 2.** The necessary and sufficient condition for (2) being true for any primitive sequence \( A \) is that \( f(B) \leq \sum 1/(p \log p) \) for any primitive sequence \( B \).

The basic idea for proving Theorem 1 is the same as that used in [7, 8]. That is, we consider the least prime factors of terms of \( A \). For a given primitive sequence \( A \) and \( m \geq 1 \), with \( A_m \) and \( A'_m \) already defined, let \( A''_m = \{ a/p_m : a \in A'_m \} \). Clearly all of them are primitive. We study upper bounds \( F_m \) of \( f(A_m) \). To find \( F_m \) we can suppose \( A_m \) is finite by Lemma 3. At first we find \( F_m \) for \( m \geq N + 1 \), where \( N = 10^5 \), by induction on \( d^0(A_m) \). It is based on the fact that

\[ f(A_m) = \sum_{i \geq m} f(A'_i) \quad \text{and} \quad f(A'_i) \leq \frac{f(A''_i)}{p_i}. \]

Then we find \( F_m \) from \( F_{m+1} \) for \( m = N, N - 1, \ldots, 2, 1 \). It is based on the fact that

\[ f(A_m) = f(A_{m+1}) + f(A'_m) \quad \text{and} \quad f(A'_m) \leq f(A''_m)/p_m. \]

At last

\[ f(A) = f(A_1) \leq F_1 < 1.84. \]

**2. Proofs of the theorems**

We have three constants: \( N = 10^5 \), \( \alpha = 1.0072629 \), and \( k = 0.0072847 \). We need five lemmas.

**Lemma 1.** We have \( p_n \geq n(\log n + \log \log n - \alpha) \) for \( n \geq 2 \) [5].

**Lemma 2.** We have

\[ \sum_{i \geq m} \frac{1}{p_i(\log i + \log \log i - k)} < \frac{1}{\log m + \log \log m - k} \]

for \( m \geq N + 1 \).

**Proof.** Put \( h(m) = \sum_{i \geq m} 1/(p_i(\log i + \log \log i - k)) \),
\[ g(m) = 1/(\log m + \log \log m - k), \]

\[ \sum_{i \geq m} \frac{1}{p_i(\log i + \log \log i - k)} < \frac{1}{\log m + \log \log m - k} \]

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c = (1 + 1/\log N)/N,

\[ H(y) = (y + \log y - \alpha)(y + \log y - k), \] and 

\[ G(y) = (y + \log y + c - k)^2. \]

Then by Lemma 1,

\[
h(m + 1) < \int_{m}^{\infty} \frac{dx}{x(\log x + \log \log x + \alpha)(\log x + \log \log x - k)} = \int_{\log m}^{\infty} \frac{dy}{H(y)},
\]

\[
g(m + 1) > \frac{1}{\log m + 1/m + \log(\log m + 1/m) - k}
\]

\[
> \frac{1}{\log m + \log \log m + (1 + 1/\log m)/m - k}
\]

\[
\geq \frac{1}{\log m + \log \log m + c - k}
\]

\[
= \int_{y = \log m}^{\infty} \frac{d(y + \log y + c - k)}{G(y)} = \int_{\log m}^{\infty} \frac{1 + 1/y}{G(y)} dy \text{ for } m \geq N.
\]

Since \( k - \alpha - 2c + 1 > 10^{-8} > 0 \),

\[
H(y)(1 + 1/y) - G(y) = (k - \alpha - 2c + 1)y + (k - \alpha - 2c + 2) \log y
\]

\[
+ (\log y - \alpha)(\log y - k)/y + ak - \alpha - k - (c - k)^2
\]

\[
> \log y + ak - \alpha - k - (c - k)^2 > 1.4 > 0
\]

for \( y \geq \log m \geq \log N \); thus \( h(m+1) < g(m+1) \) for \( m \geq N \), i.e., \( h(m) < g(m) \) for \( m \geq N + 1 \).

**Lemma 3.** If \( c \) is a constant such that \( f(S) \leq c \) for any finite subsequence \( S \) of a given primitive sequence \( A = \{a_i\} \), then \( f(A) \leq c \).

**Proof.** Put \( c_m = \sum_{1 \leq i \leq m} 1/(a_i \log a_i) \). Then \( c_m \leq c \) and \( c_m < c_{m+1} \) for \( m \geq 1 \); thus \( f(A) = \lim_{m \to \infty} c_m \leq c \).

**Lemma 4.** We have \( f(A_m) \leq 1/(\log m + \log \log m - k) \) for \( m \geq N + 1 \) for any given primitive sequence \( A = \{a_i\} \).

**Proof.** By Lemma 3, we suppose \( A_m \) is finite. Induction on \( d^0(A_m) \). If \( d^0(A_m) \leq 1 \), then

\[
f(A_m) \leq \sum_{i \geq m} \frac{1}{p_i \log p_i} < \sum_{i \geq m} \frac{1}{p_i(\log i + \log \log i - k)}
\]

\[
< \frac{1}{\log m + \log \log m - k}
\]

by Lemmas 1 and 2.

Now suppose \( d^0(A_m) = s > 1 \). Since \( A_m = \bigcup_{i \geq m} A'_i \) is disjoint, \( f(A_m) = \sum_{i \geq m} f(A'_i) \). If \( d^0(A'_i) \leq 1 \) then

\[
f(A'_i) \leq \frac{1}{p_i \log p_i} < \frac{1}{p_i(\log i + \log \log i - k)}.
\]
If $d^\circ(A'_i) > 1$ then

$$f(A'_i) < \frac{f(A''_i)}{p_i} < \frac{1}{p_i(\log i + \log \log i - k)}$$

since $d^\circ(A''_i) = d^\circ(A'_i) - 1 < s$. Thus by Lemma 2,

$$f(A_m) = \sum_{i \geq m} f(A'_i) \leq \sum_{i \geq m} \frac{1}{p_i(\log i + \log \log i - k)}$$

$$< \frac{1}{\log m + \log \log m - k}.$$

**Lemma 5.** If $f(A_{m+1}) \leq F_{m+1}$ for any primitive sequence $A$ and

$$F_m = \max\{F_{m+1}/(1 - 1/p_m), F_{m+1} + 1/(p_m \log p_m)\},$$

then $f(A_m) \leq F_m$ for any primitive sequence $A$.

**Proof.** Since $A_m = A_{m+1} \cup A'_m$ is disjoint,

$$f(A_m) = f(A_{m+1}) + f(A'_m) \leq F_{m+1} + f(A'_m).$$

By Lemma 3, we can suppose $A_m$ is finite. Induction on $d^\circ(A_m)$. If $d^\circ(A_m) < 1$ or $d^\circ(A'_m) < 1$ then

$$f(A'_m) < 1/(p_m \log p_m)$$

and

$$f(A_m) \leq F_{m+1} + 1/(p_m \log p_m) \leq F_m.$$

Now suppose $d^\circ(A_m) = s > 1$ and $d^\circ(A'_m) > 1$. Then $d^\circ(A''_m) < s$; and thus

$$f(A''_m) \leq F_m \quad \text{and} \quad f(A'_m) < f(A''_m)/p_m \leq F_m/p_m.$$

Therefore,

$$f(A_m) \leq F_{m+1} + f(A'_m) \leq F_{m+1} + F_m/p_m \leq F_m(1 - 1/p_m) + F_m/p_m = F_m.$$

**Theorem 1.** $f(A) < 1.84$ for any primitive sequence $A$.

**Proof.** Put $F_{N+1} = 1/\{\log(N+1) + \log \log(N+1) - k\}$. Then

$$f(A_{N+1}) \leq F_{N+1}$$

by Lemma 4. Let

$$F_m = \max\{F_{m+1}/(1 - 1/p_m), F_{m+1} + 1/(p_m \log p_m)\}.$$

Then

$$f(A_m) \leq F_m \quad \text{for } m = N, N - 1, \ldots, 2, 1$$

by Lemma 5. Thus $f(A) = f(A_1) \leq F_1 < 1.84$. This completes the proof.

**Theorem 2.** The necessary and sufficient condition for (2) to be true for any primitive sequence $A$ is that $f(B) \leq \sum 1/(p \log p)$ for any primitive sequence $B$.

**Proof.** (Necessity) If there exists a primitive sequence $B$ such that $f(B) > \sum 1/(p \log p)$, then there exists an integer $n$ such that

$$\sum_{b \in B, b \leq n} \frac{1}{b \log b} > \sum \frac{1}{p \log p},$$
thus
\[
\sum_{b \in B, \ b \leq n} \frac{1}{b \log b} > \sum_{p \leq n} \frac{1}{p \log p};
\]
this contradicts (2).

(Sufficiency) Suppose there exists a primitive sequence \( A = \{a_i\} \) and an integer \( n \) such that
\[
\sum_{a \in A, \ a \leq n} \frac{1}{a \log a} > \sum_{p \leq n} \frac{1}{p \log p}.
\]
Let \( B = \{a : a \in A, \ a \leq n\} \cup \{p : p > n\} \). Then \( B \) is primitive and \( f(B) > \sum 1/(p \log p) \). This is a contradiction.

Remark. From Lemma 1 and the inequality
\[
p_n \leq n(\log n + \log \log n - 0.5) \quad \text{for } n > 20 \quad [6]
\]
we can get \( 1.63 < \sum 1/(p \log p) < 1.64 \). From this fact and the above theorems we can see how far we are now from the complete proof of the conjecture.

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