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REMARK ON CERTAIN C^*-ALGEBRA EXTENSIONS CONSIDERED BY G. SKANDALIS

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ABSTRACT. Let Γ be a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one. G. Skandalis established the exact sequence

$$0 \to K(l^2(\Gamma)) \to C^*(C_\lambda^*(\Gamma), C_\rho^*(\Gamma)) \to C_\lambda^*(\Gamma \times \Gamma) \to 0.$$

In this note we give sufficient conditions under which such a short exact sequence is not semi-split. In particular, we show that such an extension has no inverse in $\operatorname{Ext}(C_\lambda^*(\Gamma \times \Gamma))$ provided that the C^*-algebra generated by the unitary representation $g \to \lambda(g)\rho(g) \otimes \lambda(g)\rho(g)$ of Γ on $l^2(\Gamma) \otimes l^2(\Gamma)$ does not contain nonzero operators from the ideal $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$.

Let G be a countable discrete group, and let λ and ρ denote the left- and right-regular representations of G on $l^2(G)$, defined by

$$(\lambda(g)\xi)(t) = \xi(g^{-1}t) \quad \text{and} \quad (\rho(g)\xi)(t) = \xi(tg) \qquad (\xi \in l^2(G)).$$

The C^*-algebras $C_\lambda^*(G)$ and $C_\rho^*(G)$, generated respectively by $\lambda(G)$ and $\rho(G)$, are commuting and $C_\rho^*(G) = JC_\lambda^*(G)J$, where J is the involution on $l^2(G)$ defined by $(J\xi)(t) = \xi(t^{-1})$. Let $C_{\lambda,\rho}^*(G)$ denote the C^*-algebra generated by $C_\lambda^*(G) \cup C_\rho^*(G)$.

If G satisfies the infinite conjugacy class (ICC) condition, then the map given by $\sum_i x_i \odot y_i \to \sum_i x_i J y_i J$ is an isomorphism of the algebraic tensor product $C_\lambda^*(G) \odot C_\lambda^*(G)$ onto a dense *-subalgebra of $C_{\lambda,\rho}^*(G)$ [8, IV, 4.20]. This defines a C^*-norm $\| \ \|_{\lambda,\rho}$ on $C_\lambda^*(G) \odot C_\lambda^*(G)$ via $\|\sum_i x_i \odot y_i\|_{\lambda,\rho} = \|\sum_i x_i J y_i J\|$; and thus $C_{\lambda,\rho}^*(G)$ is a faithful representation of the C^*-tensor product $C_\lambda^*(G) \otimes_{\lambda,\rho} C_\lambda^*(G)$—the closure of $C_\lambda^*(G) \odot C_\lambda^*(G)$ in the norm $\| \ \|_{\lambda,\rho}$. It is known that the norm $\| \ \|_{\lambda,\rho}$ dominates the spatial (minimal) norm on $C_\lambda^*(G) \odot C_\lambda^*(G)$ (and is equal to the spatial norm when G is amenable). Hence there is a canonical surjective *-homomorphism $\varphi\colon C_{\lambda,\rho}^*(G) \to C_\lambda^*(G) \otimes_{\min} C_\lambda^*(G) = C_\lambda^*(G \times G)$.

In [7] Skandalis showed that if Γ is a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one, then $C_{\lambda,\rho}^*(\Gamma)$ contains $K(l^2(\Gamma))$, the algebra of compact operators on $l^2(\Gamma)$, and the following exact sequence

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holds:

$$(*) \qquad 0 \to K(l^2(\Gamma)) \to C^*_{\lambda,\rho}(\Gamma) \xrightarrow{\varphi} C^*_\lambda(\Gamma \times \Gamma) \to 0.$$

This generalizes the earlier result of Akemann and Ostrand [1] obtained for $\Gamma = F_2$, the free group on two generators.

The following theorem provides a sufficient condition under which an extension $(*)$ has no inverse in $\operatorname{Ext}(C^*_\lambda(\Gamma \times \Gamma))$.

Theorem. *Let Γ be a discrete (nonamenable, ICC) group satisfying $(*)$. Suppose that for some $g_i \in \Gamma$ and $c_i \in \mathbb{C}$ $(i = 1, \dots, N)$:*

(i) *the C^*-algebra $C^*(\sum_{i=1}^n c_i\lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i))$, generated by the operator $\sum_{i=1}^n c_i\lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i)$ on $l^2(\Gamma) \otimes l^2(\Gamma)$, does not contain nonzero operators from the ideal $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$ in $B(l^2(\Gamma)) \otimes B(l^2(\Gamma))$; and*

(ii) *the norm $\|\sum_{i=1}^n c_i\lambda(g_i \times g_i)\rho(g_i \times g_i)\|$ is not equal to the norm $\|\sum_{i=1}^n c_i\lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i)\|$ of the operator $\sum_{i=1}^n c_i\lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i)$ on $l^2(\Gamma \times \Gamma) \otimes l^2(\Gamma \times \Gamma)$. (In other words, for $x = \sum_{i=1}^n c_i\lambda(g_i \times g_i) \odot \lambda(g_i \times g_i) \in C^*_\lambda(\Gamma \times \Gamma) \odot C^*_\lambda(\Gamma \times \Gamma)$, $\|x\|_{\min} \neq \|x\|_{\lambda,\rho}$).*

Then the embedding of the algebra $C^*_\lambda(\Gamma \times \Gamma)$ into the Calkin algebra $B(l^2(\Gamma))/K(l^2(\Gamma))$, determined by $(*)$, does not have a completely positive lifting to $B(l^2(\Gamma))$.

Proof. Let $\sigma\colon C^*_\lambda(\Gamma \times \Gamma) \to B(l^2(\Gamma))/K(l^2(\Gamma))$ denote the embedding determined by $(*)$. Suppose $\psi\colon C^*_\lambda(\Gamma \times \Gamma) \to B(l^2(\Gamma))$ is a completely positive lifting of σ. Then we have the following commutative diagram:

$$\begin{tikzcd}
 & C^*_{\lambda,\rho}(\Gamma) \arrow[d, "\varphi"] \\
C^*_\lambda(\Gamma \times \Gamma) \arrow[ur, "\Psi"] \arrow[r, "\sigma"] & B(l^2(\Gamma))/K(l^2(\Gamma))
\end{tikzcd}$$

In particular, for $g \in \Gamma$, $\psi(\lambda(g \times g)) = \lambda(g)\rho(g) + k$ for some $k \in K(l^2(\Gamma))$ since $\varphi(\lambda(g)\rho(g)) = \sigma(\lambda(g \times g))$. We can assume that ψ is unital [2, Lemma 3.3]. From Stinespring's theorem, $\psi(\cdot) = U^*\pi(\cdot)U$ for some representation π of $C^*_\lambda(\Gamma \times \Gamma)$ on a Hilbert space $\mathscr{H}$ and isometry $U\colon l^2(\Gamma) \to \mathscr{H}$. Let $\pi \otimes \pi$ denote the tensor product representation of $C^*_\lambda(\Gamma \times \Gamma) \otimes_{\min} C^*_\lambda(\Gamma \times \Gamma)$ on $\mathscr{H} \otimes \mathscr{H}$. Then the map $\psi \otimes \psi\colon C^*_\lambda(\Gamma \times \Gamma) \otimes_{\min} C^*_\lambda(\Gamma \times \Gamma) \to C^*_{\lambda,\rho}(\Gamma) \otimes_{\min} C^*_{\lambda,\rho}(\Gamma)$, defined by $\psi \otimes \psi(\cdot) = (U \otimes U)^*(\pi \otimes \pi)(\cdot)(U \otimes U)$, is contractive (completely positive) and satisfies

$$\psi \otimes \psi(x \otimes y) = \psi(x) \otimes \psi(y) \quad \text{for all } x, y \in C^*_\lambda(\Gamma \times \Gamma).$$

Therefore, in particular,

$$\psi \otimes \psi\left(\sum_{i=1}^n c_i\lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i)\right) = \sum_{i=1}^n c_i\lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) + j_0$$

for some $j_0 \in K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$, and

$$(1) \qquad \left\|\sum_{i=1}^n c_i\lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) + j_0\right\| \le \left\|\sum_{i=1}^n c_i\lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i)\right\|.$$

Denote

$$a = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i)\rho(g_i \times g_i),$$

$$b = \sum_{i=1}^{n} c_i \lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i)$$

$$\left(= \sum_{i=1}^{n} c_i (\lambda(g_i) \otimes \lambda(g_i))(\rho(g_i) \otimes \rho(g_i)) \right),$$

and

$$J = K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma)).$$

The operators a and b are unitarily equivalent via the natural unitary transformation $V\colon l^2(\Gamma) \otimes l^2(\Gamma) \to l^2(\Gamma \times \Gamma)$, and so are the C^*-algebras $C^*(a)$ and $C^*(b)$, generated respectively by a and b. Since J is a closed two-sided ideal of $B(l^2(\Gamma)) \otimes B(l^2(\Gamma))$ and $C^*(b) \cap J = (0)$, we can define a C^*-algebra homomorphism $\theta\colon C^*(b) + J \to C^*(a)$ by

$$\theta(z + j) = VzV^* \qquad (z \in C^*(b),\ j \in J).$$

In particular, since $\theta(b + j_0) = VbV^* = a$, we have $\|a\| \le \|b + j_0\|$. Combining this with (1), we get

$$\left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i)\rho(g_i \times g_i) \right\| \le \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\|.$$

Consequently, for $x = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \odot \lambda(g_i \times g_i) \in C_\lambda^*(\Gamma \times \Gamma) \odot C_\lambda^*(\Gamma \times \Gamma)$ we have $\|x\|_{\lambda,\rho} \le \|x\|_{\min}$. This implies $\|x\|_{\lambda,\rho} = \|x\|_{\min}$, which contradicts the hypothesis.

Corollary. *Let Γ be a discrete (nonamenable, ICC) group satisfying $(*)$. Let μ be the unitary representation of Γ on $l^2(\Gamma) \otimes l^2(\Gamma)$ defined by $\mu(g) = \lambda(g)\rho(g) \otimes \lambda(g)\rho(g)$, and let $C_\mu^*(\Gamma)$ be the C^*-algebra generated by μ. If $C_\mu^*(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0)$, then the extension (class of) $(*)$ has no inverse in* $\operatorname{Ext}(C_\lambda^*(\Gamma \times \Gamma))$.

Proof. The representation μ contains the trivial representation of Γ, since $\mu(g)(\xi_0 \otimes \xi_0) = \xi_0 \otimes \xi_0$ for all g, where ξ_0 denotes a function supported by the identity of Γ. On the other hand, the representation of Γ on $l^2(\Gamma \times \Gamma) \otimes l^2(\Gamma \times \Gamma)$, defined by $\delta(g) = \lambda(g \times g) \otimes \lambda(g \times g)$, is equivalent to a multiple of the left-regular representation of Γ [4, 13.11.3]. Since Γ is nonamenable, the map $\lambda(g \times g) \otimes \lambda(g \times g) \to \lambda(g)\rho(g) \otimes \lambda(g)\rho(g)$ does not extend to a $*$-homomorphism of $C_\delta^*(\Gamma)$ onto $C_\mu^*(\Gamma)$. Therefore, there are $g_i \in \Gamma$ and $c_i \in \mathbb{C}$ $(i = 1, \dots, n)$ such that

$$\left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\| < \left\| \sum_{i=1}^{n} c_i \lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) \right\|.$$

As observed earlier,

$$\left\| \sum_{i=1}^{n} c_i \lambda(g_i)\rho(g_i) \otimes \lambda(g_i)\rho(g_i) \right\| = \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i)\rho(g_i \times g_i) \right\|;$$

so that the conclusion follows from the preceding theorem.

Remark. The hypothesis of the corollary is reminiscent of, but is much weaker than, the inner amenability of $\Gamma \times \Gamma$ (see [5, 6, 3]). Let α and β denote the unitary representations of $\Gamma \times \Gamma$ and Γ on $l^2(\Gamma \times \Gamma)$, given by $\alpha(h) = \lambda(h)\rho(h)$ and $\beta(g) = \lambda(g \times g)\rho(g \times g)$. The representation μ of the corollary is unitarily equivalent to β; and $\beta = \alpha \circ \Delta$, where Δ denotes the diagonal map of Γ into $\Gamma \times \Gamma$. If Γ is an ICC group, the inner amenability of $\Gamma \times \Gamma$ is equivalent to $C^*_\alpha(\Gamma) \cap K(l^2(\Gamma \times \Gamma)) = (0)$ [3]. In particular, in this case $C^*_\mu(\Gamma) \cap K(l^2(\Gamma) \otimes l^2(\Gamma)) = (0)$, and so also

$$C^*_\mu(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0),$$

since by [9] $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$ is a proper ideal of $K(l^2(\Gamma) \otimes l^2(\Gamma))$.

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