REMARK ON CERTAIN $C^*$-ALGEBRA EXTENSIONS CONSIDERED BY G. SKANDALIS

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ABSTRACT. Let $\Gamma$ be a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one. G. Skandalis established the exact sequence

$$0 \to K(l^2(\Gamma)) \to C^*(C^*_\lambda(\Gamma), C^*_\rho(\Gamma)) \to C^*_\lambda(\Gamma \times \Gamma) \to 0.$$ 

In this note we give sufficient conditions under which such a short exact sequence is not semi-split. In particular, we show that such an extension has no inverse in $\text{Ext}(C^*_\lambda(\Gamma \times \Gamma))$ provided that the $C^*$-algebra generated by the unitary representation $g \to \lambda(g) \rho(g) \otimes \delta(g) \rho(g)$ of $\Gamma$ on $l^2(\Gamma) \otimes l^2(\Gamma)$ does not contain nonzero operators from the ideal $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$.

Let $G$ be a countable discrete group, and let $\lambda$ and $\rho$ denote the left- and right-regular representations of $G$ on $l^2(G)$, defined by

$$(\lambda(g)\xi)(t) = \xi(g^{-1}t) \quad \text{and} \quad (\rho(g)\xi)(t) = \xi(tg) \quad (\xi \in l^2(G)).$$

The $C^*$-algebras $C^*_\lambda(G)$ and $C^*_\rho(G)$, generated respectively by $\lambda(G)$ and $\rho(G)$, are commuting and $C^*_\rho(G) = JC^*_\lambda(G)J$, where $J$ is the involution on $l^2(G)$ defined by $(J\xi)(t) = \xi(t^{-1})$. Let $C^*_{\lambda, \rho}(G)$ denote the $C^*$-algebra generated by $C^*_\lambda(G) \cup C^*_\rho(G)$.

If $G$ satisfies the infinite conjugacy class (ICC) condition, then the map given by $\sum_i x_i \otimes y_i \mapsto \sum_i x_i J y_i J$ is an isomorphism of the algebraic tensor product $C^*_\lambda(G) \otimes C^*_\rho(G)$ onto a dense $*$-subalgebra of $C^*_{\lambda, \rho}(G)$ [8, IV, 4.20]. This defines a $C^*$-norm $\| \|_{\lambda, \rho}$ on $C^*_\lambda(G) \otimes C^*_\rho(G)$ via $\| \sum_i x_i \otimes y_i \|_{\lambda, \rho} = \| \sum_i x_i J y_i J \|$; and thus $C^*_{\lambda, \rho}(G)$ is a faithful representation of the $C^*$-tensor product $C^*_\lambda(G) \otimes_{\lambda, \rho} C^*_\rho(G)$—the closure of $C^*_\lambda(G) \otimes C^*_\rho(G)$ in the norm $\| \|_{\lambda, \rho}$. It is known that the norm $\| \|_{\lambda, \rho}$ dominates the spatial (minimal) norm on $C^*_\lambda(G) \otimes C^*_\rho(G)$ (and is equal to the spatial norm when $G$ is amenable). Hence there is a canonical surjective $*$-homomorphism $\varphi: C^*_{\lambda, \rho}(G) \to C^*_\lambda(G) \otimes_{\text{min}} C^*_\rho(G) = C^*_\rho(G \times G)$.

In [7] Skandalis showed that if $\Gamma$ is a nonamenable, discrete ICC subgroup of a connected simple Lie group of real-rank one, then $C^*_\lambda(\Gamma \times \Gamma)$ contains $K(l^2(\Gamma))$, the algebra of compact operators on $l^2(\Gamma)$, and the following exact sequence

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holds:

\((*)\) \quad 0 \rightarrow K(l^2(\Gamma)) \rightarrow C^*_{\lambda, \rho}(\Gamma) \overset{\varphi}{\rightarrow} C^*_\lambda(\Gamma \times \Gamma) \rightarrow 0.

This generalizes the earlier result of Akemann and Ostrand [1] obtained for \(\Gamma = F_2\), the free group on two generators.

The following theorem provides a sufficient condition under which an extension \((*)\) has no inverse in \(\text{Ext}(C^*_\lambda(\Gamma \times \Gamma))\).

**Theorem.** Let \(\Gamma\) be a discrete (nonamenable, ICC) group satisfying \((*)\). Suppose that for some \(g_i \in \Gamma\) and \(c_i \in \mathbb{C}\) (\(i = 1, \ldots, N\)):

(i) the \(C^*\)-algebra \(C^*(\sum_{i=1}^n c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i))\), generated by the operator \(\sum_{i=1}^n c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i)\) on \(l^2(\Gamma) \otimes l^2(\Gamma)\), does not contain nonzero operators from the ideal \(K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))\) in \(B(l^2(\Gamma)) \otimes B(l^2(\Gamma))\); and

(ii) the norm \(\| \sum_{i=1}^n c_i \lambda(g_i x g_i) \rho(g_i x g_i)\|\) is not equal to the norm \(\| \sum_{i=1}^n c_i \lambda(g_i x g_i) \otimes \lambda(g_i x g_i)\|\) of the operator \(\sum_{i=1}^n c_i \lambda(g_i x g_i) \otimes \lambda(g_i x g_i)\) on \(l^2(\Gamma \times \Gamma) \otimes l^2(\Gamma \times \Gamma)\). (In other words, for \(x = \sum_{i=1}^n c_i \lambda(g_i x g_i) \otimes \lambda(g_i x g_i) \in C^*_\lambda(\Gamma \times \Gamma) \otimes C^*_\lambda(\Gamma \times \Gamma), \|x\|_{\min} \neq \|x\|_{\lambda, \rho}\).

Then the embedding of the algebra \(C^*_\lambda(\Gamma \times \Gamma)\) into the Calkin algebra \(B(l^2(\Gamma))/K(l^2(\Gamma))\), determined by \((*)\), does not have a completely positive lifting to \(B(l^2(\Gamma))\).

**Proof.** Let \(\sigma: C^*_\lambda(\Gamma \times \Gamma) \rightarrow B(l^2(\Gamma))/K(l^2(\Gamma))\) denote the embedding determined by \((*)\). Suppose \(\psi: C^*_\lambda(\Gamma \times \Gamma) \rightarrow B(l^2(\Gamma))\) is a completely positive lifting of \(\sigma\). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
C^*_\lambda(\Gamma \times \Gamma) & \xrightarrow{\psi} & B(l^2(\Gamma))/K(l^2(\Gamma)) \\
\downarrow \sigma & & \\
C^*_\lambda(\Gamma \times \Gamma) & \xrightarrow{\varphi} & B(l^2(\Gamma))/K(l^2(\Gamma))
\end{array}
\]

In particular, for \(g \in \Gamma\), \(\psi(\lambda(g x g)) = \lambda(g) \rho(g) + k\) for some \(k \in K(l^2(\Gamma))\) since \(\varphi(\lambda(g) \rho(g)) = \sigma(\lambda(g x g))\). We can assume that \(\psi\) is unital [2, Lemma 3.3]. From Stinespring's theorem, \(\psi(\cdot) = U^* \pi(\cdot) U\) for some representation \(\pi\) of \(C^*_\lambda(\Gamma \times \Gamma)\) on a Hilbert space \(\mathcal{H}\) and isometry \(U: l^2(\Gamma) \rightarrow \mathcal{H}\). Let \(\pi \otimes \pi\) denote the tensor product representation of \(C^*_\lambda(\Gamma \times \Gamma) \otimes_{\min} C^*_\lambda(\Gamma \times \Gamma)\) on \(\mathcal{H} \otimes \mathcal{H}\). Then the map \(\psi \otimes \psi: C^*_\lambda(\Gamma \times \Gamma) \otimes_{\min} C^*_\lambda(\Gamma \times \Gamma) \rightarrow C^*_\lambda, \rho(\Gamma \times \Gamma) \otimes_{\min} C^*_\lambda, \rho(\Gamma \times \Gamma)\), defined by \(\psi \otimes \psi(\cdot) = (U \otimes U)^* (\pi \otimes \pi)(\cdot) (U \otimes U)\), is contractive (completely positive) and satisfies

\(\psi \otimes \psi(x \otimes y) = \psi(x) \otimes \psi(y)\) for all \(x, y \in C^*_\lambda(\Gamma \times \Gamma)\).

Therefore, in particular,

\[
\psi \otimes \psi \left( \sum_{i=1}^n c_i \lambda(g_i x g_i) \otimes \lambda(g_i x g_i) \right) = \sum_{i=1}^n c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) + j_0
\]

for some \(j_0 \in K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))\), and

\[
(1) \quad \left\| \sum_{i=1}^n c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) + j_0 \right\| \leq \left\| \sum_{i=1}^n c_i \lambda(g_i x g_i) \otimes \lambda(g_i x g_i) \right\|.
\]
Denote
\[ a = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i), \]
\[ b = \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \]
\[ \left( = \sum_{i=1}^{n} c_i (\lambda(g_i) \otimes \lambda(g_i)) \rho(g_i) \otimes \rho(g_i) \right), \]
and
\[ J = K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma)). \]
The operators \( a \) and \( b \) are unitarily equivalent via the natural unitary transformation \( V: l^2(\Gamma) \otimes l^2(\Gamma) \to l^2(\Gamma \times \Gamma) \), and so are the \( C^* \)-algebras \( C^*(a) \) and \( C^*(b) \), generated respectively by \( a \) and \( b \). Since \( J \) is a closed two-sided ideal of \( B(l^2(\Gamma)) \otimes B(l^2(\Gamma)) \) and \( C^*(b) \cap J = (0) \), we can define a \( C^* \)-algebra homomorphism \( \theta: C^*(b) + J \to C^*(a) \) by
\[ \theta(z + j) = VzV^* \quad (z \in C^*(b), \ j \in J). \]
In particular, since \( \theta(b + j_0) = VbV^* = a \), we have \( \|a\| \leq \|b + j_0\| \). Combining this with (1), we get
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i) \right\| \leq \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\|. \]
Consequently, for \( x = \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \in C^*_\lambda(\Gamma \times \Gamma) \otimes C^*_\lambda(\Gamma \times \Gamma) \) we have \( \|x\|_{\lambda, \rho} \leq \|x\|_{\min} \). This implies \( \|x\|_{\lambda, \rho} = \|x\|_{\min} \), which contradicts the hypothesis.

**Corollary.** Let \( \Gamma \) be a discrete (nonamenable, ICC) group satisfying (*). Let \( \mu \) be the unitary representation of \( \Gamma \) on \( l^2(\Gamma) \otimes l^2(\Gamma) \) defined by \( \mu(g) = \lambda(g) \rho(g) \otimes \lambda(g) \rho(g) \), and let \( C^*_\mu(\Gamma) \) be the \( C^* \)-algebra generated by \( \mu \). If \( C^*_\mu(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0), \) then the extension (class of) (*) has no inverse in \( \text{Ext}(C^*_\lambda(\Gamma \times \Gamma)) \).

**Proof.** The representation \( \mu \) contains the trivial representation of \( \Gamma \), since \( \mu(g)(\xi_0 \otimes \xi_0) = \xi_0 \otimes \xi_0 \) for all \( g \), where \( \xi_0 \) denotes a function supported by the identity of \( \Gamma \). On the other hand, the representation of \( \Gamma \) on \( l^2(\Gamma \times \Gamma) \otimes l^2(\Gamma \times \Gamma) \), defined by \( \delta(g) = \lambda(g \times g) \otimes \lambda(g \times g) \), is equivalent to a multiple of the left-regular representation of \( \Gamma \) \([4, 13.11.3]\). Since \( \Gamma \) is nonamenable, the map \( \lambda(g \times g) \otimes \lambda(g \times g) \to \lambda(g) \rho(g) \otimes \lambda(g) \rho(g) \) does not extend to a \( * \)-homomorphism of \( C^*_\delta(\Gamma) \) onto \( C^*_\mu(\Gamma) \). Therefore, there are \( g_i \in \Gamma \) and \( c_i \in \mathbb{C} \) \((i = 1, \ldots, n)\) such that
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \otimes \lambda(g_i \times g_i) \right\| < \left\| \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \right\|. \]
As observed earlier,
\[ \left\| \sum_{i=1}^{n} c_i \lambda(g_i) \rho(g_i) \otimes \lambda(g_i) \rho(g_i) \right\| = \left\| \sum_{i=1}^{n} c_i \lambda(g_i \times g_i) \rho(g_i \times g_i) \right\|. \]
so that the conclusion follows from the preceding theorem.

Remark. The hypothesis of the corollary is reminiscent of, but is much weaker than, the inner amenability of $\Gamma \times \Gamma$ (see [5, 6, 3]). Let $\alpha$ and $\beta$ denote the unitary representations of $\Gamma \times \Gamma$ and $\Gamma$ on $l^2(\Gamma \times \Gamma)$, given by $\alpha(h) = \lambda(h)\rho(h)$ and $\beta(g) = \lambda(g \times g)\rho(g \times g)$. The representation $\mu$ of the corollary is unitarily equivalent to $\beta$; and $\beta = \alpha \circ \Delta$, where $\Delta$ denotes the diagonal map of $\Gamma$ into $\Gamma \times \Gamma$. If $\Gamma$ is an ICC group, the inner amenability of $\Gamma \times \Gamma$ is equivalent to $C^*_\mu(\Gamma) \cap K(l^2(\Gamma \times \Gamma)) = (0)$ [3]. In particular, in this case $C^*_\mu(\Gamma) \cap K(l^2(\Gamma) \otimes l^2(\Gamma)) = (0)$, and so also

$$C^*_\mu(\Gamma) \cap (K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))) = (0),$$

since by [9] $K(l^2(\Gamma)) \otimes B(l^2(\Gamma)) + B(l^2(\Gamma)) \otimes K(l^2(\Gamma))$ is a proper ideal of $K(l^2(\Gamma) \otimes l^2(\Gamma))$.

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