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BOUNDED APPROXIMATE IDENTITIES IN THE ALGEBRA OF COMPACT OPERATORS ON A BANACH SPACE

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ABSTRACT. We characterize Banach spaces X for which the algebra of compact operators $\mathscr{K}(X)$ admits approximate bounded right identities.

INTRODUCTION

In his monograph [J] Johnson asked if, for every Banach space X, the algebra $\mathscr{K}(X)$ of all compact operators on X is amenable. It is known that an amenable Banach algebra admits bounded approximate identities and that there exists a Banach space X such that the algebra $\mathscr{K}(X)$ does not admit any bounded left approximate identity [D]. Therefore, the answer to Johnson's question is no. It is interesting to study connections between amenability of the algebra $\mathscr{K}(X)$ and property of structure of X. In this paper we characterize Banach spaces X for which $\mathscr{K}(X)$ admits bounded right approximate identities. We show that if $\mathscr{K}(X)$ admits a bounded right approximate identity then it admits a bounded left approximate identity. An example of a Banach space X for which $\mathscr{K}(X)$ admits a bounded left approximate identity and no bounded right approximate identity is given.

NOTATION AND PRELIMINARIES

If X is a Banach space, $\mathscr{K}(X)$ will denote the algebra of all compact operators on X and B_X will denote the closed unit ball of X. All the operators are assumed to be continuous.

Let $\lambda \geq 1$. A Banach space X is said to have the λ-bounded compact approximation property (λ-b.c.a.p. in short) if for every finite subset $F \subset X$ and for every $\varepsilon > 0$ there is an operator $u \in \mathscr{K}(X)$ such that $\|u\| \leq \lambda$ and $\|u(x) - x\| \leq \varepsilon$ for all $x \in F$. The space is said to have the bounded compact approximation property (b.c.a.p. in short) if it has the λ-b.c.a.p. for a $\lambda \geq 1$.

Let $\lambda \geq 1$. The dual X^* of a Banach space X is said to have the $*\lambda$-bounded compact approximation property ($*\lambda$-b.c.a.p. in short) if for every finite subset $F \subset X^*$ and for every $\varepsilon > 0$ there is an operator $u \in \mathscr{K}(X)$ such that $\|u\| \leq \lambda$

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and $\|u^*(x^*)-x^*\| \leq \varepsilon$ for all $x^* \in F$. The space is said to have the $*$-bounded compact approximation property ($*$-b.c.a.p. in short) if it has the $*\lambda$-b.c.a.p. for a $\lambda \geq 1$.

A net $(u_\alpha)_\alpha$ of a Banach algebra A is a bounded right (left) approximate identity if $\sup_\alpha\|u_\alpha\| < +\infty$ and $\lim_\alpha au_\alpha = a$ ($\lim_\alpha u_\alpha a = a$) for every $a \in A$. A net which is a bounded left approximate identity and a bounded right approximate identity is a bounded approximate identity.

Bounded approximate identities

Theorem 1. *If X is a Banach space such that X^* has the $*$-b.c.a.p. then X has the b.c.a.p.*

Proof. Let $\lambda \geq 1$ such that X^* has the $*\lambda$-b.c.a.p.; there exists a net $(u_\alpha)_{\alpha\in\mathscr{A}}$ of $\mathscr{K}(X)$ such that $\sup_\alpha\|u_\alpha\| \leq \lambda$ and $\lim_\alpha\|u_\alpha^*(x^*)-x^*\| = 0$ for every $x^* \in X^*$. Then we have $u_\alpha(x) \to x$ weakly for every $x \in X$. Let F be a finite subset of X and $\varepsilon > 0$. There exists a convex combination v of the $\{u_\alpha\,;\,\alpha \in \mathscr{A}\}$ such that $\|v(x) - x\| \leq \varepsilon$ for every $x \in F$; of course, $\|v\| \leq \lambda$ and $v \in \mathscr{K}(X)$.

Remark 1. Let X be a reflexive Banach space. Then the following properties are equivalent:

(a) X has the b.c.a.p.,
(b) X^* has the b.c.a.p.,
(c) X^* has has the $*$-b.c.a.p.

Remark 2. There exists a Banach space Z with the b.c.a.p. such that Z^* does not have the $*$-b.c.a.p. We know that there exists a separable reflexive Banach space Y without the b.c.a.p. [LT]. Using the result of Lindenstrauss [L] there exists a Banach space Z with a boundedly complete basis such that Z^* has a complemented subspace isomorphic to Y^*. Every complemented subspace of a Banach space with the b.c.a.p. has the b.c.a.p., and so Z^* does not have the b.c.a.p.; it is then obvious that Z^* does not have the $*$-b.c.a.p.

Theorem 2. *Let X be a Banach space. Then $\mathscr{K}(X)$ admits a bounded right approximate identity if and only if X^* has the $*$-b.c.a.p.*

Proof. (i) Let X be a Banach space such that $\mathscr{K}(X)$ admits a bounded right approximate identity $(v_\beta)_{\beta\in\mathscr{B}}$ and $\sup_\beta\|v_\beta\| \leq \lambda$. Let F be a finite subset of X^*. Then there exists a projection on the finite-dimensional subspace spanned by F. According to Lemma 3.1 of [JRZ] there exists a finite rank projection π on X such that $\pi^*(x^*) = x^*$ for every $x^* \in F$. We have $\lim_\beta\|\pi \circ v_\beta - \pi\| = 0$ so $\lim_\beta\|v_\beta^* \circ \pi^* - \pi^*\| = 0$ and then for every $x^* \in F$ we have

$$x^* = \pi^*(x^*) = \lim_\beta v_\beta^*(\pi^*(x^*)) = \lim_\beta v_\beta^*(x^*).$$

(ii) Let X be a Banach space such that X^* has the $*\lambda$-b.c.a.p. There exists a net $(u_\alpha)_{\alpha\in\mathscr{A}}$ on $\mathscr{K}(X)$ such that $\sup_\alpha\|u_\alpha\| \leq \lambda$ and $\lim_\alpha\|u_\alpha^*(x^*)-x^*\| = 0$ for every $x^* \in X^*$. We shall show that $(u_\alpha)_{\alpha\in\mathscr{A}}$ is a bounded right approximate identity on $\mathscr{K}(X)$. Let $u \in \mathscr{K}(X)$, $\varepsilon > 0$, and $\varepsilon' > 0$ such that $(\lambda+3)\varepsilon' \leq \varepsilon$. u^* is a compact operator on X^*; then $u^*(B_{X^*})$ is a totally bounded subset of X^*. There exists a finite subset $F \subset u^*(B_{X^*})$ such that

$$\min\{\|u^*(x^*) - y^*\|\,;\, y^* \in F\} \leq \varepsilon'$$

for every $x^* \in B_{X^*}$. There exists $\alpha_0 \in \mathscr{A}$ such that $\|u_\alpha^*(y^*) - y^*\| \le \varepsilon'$ for every $\alpha \succeq \alpha_0$ and for every $y^* \in F$. Let $x^* \in B_{X^*}$ and $y^* \in F$ such that $\|u^*(x^*) - y^*\| \le \varepsilon'$. For $\alpha \succeq \alpha_0$ we have

$$\begin{aligned}\|u^*(x^*) - u_\alpha^* \circ u^*(x^*)\| &\le \|u^*(x^*) - y^*\| + \|y^* - u_\alpha^*(y^*)\| + \|u_\alpha(y^* - u^*(x^*))\| \\ &\le \varepsilon' + \varepsilon' + \lambda\varepsilon' = (\lambda + 3)\varepsilon' \le \varepsilon,\end{aligned}$$

and so $\|u^* - u_\alpha^* \circ u^*\| = \|u - u \circ u_\alpha\| \le \varepsilon$.

Remark 3. Dixon has shown in [D] that a Banach space X has the b.c.a.p. if and only if $\mathscr{K}(X)$ admits a bounded left approximate identity. Using Theorems 1 and 2 we deduce that $\mathscr{K}(X)$ admits a bounded left approximate identity if $\mathscr{K}(X)$ admits a bounded right approximate identity.

Remark 4. The algebra $\mathscr{K}(Z)$ of compact operators on the space Z introduced in Remark 2 admits a bounded left approximate identity and no bounded right approximate identity.

Remark 5. Let X be a reflexive Banach space; $\mathscr{K}(X)$ admits a bounded left approximate identity if and only if $\mathscr{K}(X)$ admits a bounded right approximate identity; $\mathscr{K}(X)$ admits a bounded approximate identity if and only if $\mathscr{K}(X^*)$ admits a bounded approximate identity.

Remark 6. Let X be a Banach space such that $\mathscr{K}(X)$ is an M-ideal in the algebra of all operators on X. Then $\mathscr{K}(X)$ admits a bounded approximate identity $(T_\alpha)_\alpha$ such that $\sup_\alpha\|T_\alpha\| \le 1$ and $\lim_\alpha\|I - T_\alpha\| = 1$ [HL].

Open problems

The author expresses his gratitude to the referee for his valuable open problems.

Let us recall that a Banach space X is said to have the bounded approximation property (b.a.p. in short) if there exists $\lambda \ge 1$ such that for every $\varepsilon > 0$ and for every finite subset $F \subset X$ there exists a finite rank operator u in X such that $\|u\| \le \lambda$ and $\|u(x) - x\| \le \varepsilon$ for all $x \in F$.

Question 1. If X has the b.c.a.p., does X have the b.a.p.? At least what additional property of X is needed to ensure that the b.c.a.p. implies the b.a.p.?

Question 2. If X^* has the b.c.a.p., does X^* have the $*$-b.c.a.p., or at least, does X have the b.c.a.p.? Of course if the first part of Question 1 has an affirmative answer then so does the second part of Question 2.

A Banach space X is an Asplund space if every separable subspace of X has a separable dual. It is known [G] that if X is reflexive (or more generally, the dual of an Asplund space) and has the b.a.p. then it has the 1-b.a.p.

Question 3. If X is reflexive (or more generally, the dual of an Asplund space) and has the b.c.a.p, does X have the 1-b.c.a.p.?

References

[D] P. G. Dixon, *Left approximate identities in algebras of compact operators on Banach spaces*, Proc. Roy. Soc. Edinburgh Sect. A **104** (1986), 169–175.

[G] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., no. 16, Amer. Math. Soc., Providence, RI, 1955.

[J] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., no. 127, Amer. Math. Soc., Providence, RI, 1972.

[JRZ] W. B. Johnson, H. P. Rosenthal, and M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. **9** (1971), 488–506.

[HL] P. Harmand and A. Lima, *Banach spaces which are M-ideals in their biduals*, Trans. Amer. Math. Soc. **283** (1984), 253–264.

[L] J. Lindenstrauss, *On James's paper "Separable conjugate spaces"*, Israel J. Math. **9** (1971), 279–284.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*. I, Springer-Verlag, Berlin and New York, 1977.

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