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LIFTINGS AND THE PROPERTY OF BAIRE IN LOCALLY COMPACT GROUPS

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ABSTRACT. For each locally compact group G with Haar measure μ, we obtain the following results. The first is a version for group quotients of a classical result of Kuratowski and Ulam on first category subsets of the plane. The second is a strengthening of a theorem of Kupka and Prikry; we obtain it by a much simpler technique, building on work of Talagrand and Losert.

Theorem 1. *If G is σ-compact, $H \subseteq G$ is a closed normal subgroup, and $\pi: G \to G/H$ is the usual projection, then for each first category set $A \subseteq G$, there is a first category set $E \subseteq G/H$ such that for each $y \in (G/H) - E$, $A \cap \pi^{-1}(y)$ is a first category set relative to $\pi^{-1}(y)$.*

Theorem 2. *If G is not discrete, then there is a Borel set $E \subseteq G$ such that for any translation-invariant lifting ρ for (G, μ), $\rho(E)$ is not universally measurable and does not have the Baire property.*

0. INTRODUCTION

A *lifting* for a measure space (X, μ) is a Boolean homomorphism ρ from the algebra Σ of measurable sets into itself, whose kernel is the ideal of sets of measure zero, and for which $\rho(E) \Delta E$ has measure zero for each $E \in \Sigma$. If X is a topological space and the domain of μ includes the open sets, then ρ is called a *strong* lifting if $U \subseteq \rho(U)$ for each open set U. An equivalent way of giving a lifting is to give a *lifting for* $\mathscr{L}^\infty(X, \mu)$, i.e., an algebra homomorphism ρ from $\mathscr{L}^\infty(X, \mu)$ into itself for which $\rho(f) = f$ a.e. whenever $f \in \mathscr{L}^\infty(X, \mu)$. See [I2] for the equivalence.

Every complete σ-finite measure space has a lifting [M], every locally compact group has a *translation-invariant* lifting for its Haar measure (i.e., $\rho(gE) = g\rho(E)$ for each element g of the group and for each measurable set E), and every translation-invariant lifting is strong [I1]. [For each group concept that has a 'left' and a 'right' version (e.g., translation, coset, Haar measure), the 'left' version is intended in this paper.]

The constructions of liftings are not effective. The necessity of their noneffectiveness has been demonstrated in several ways. The existence of a lifting ρ

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for Lebesgue measure on $\mathbf{R}$ implies in $ZF + DC$ the existence of a nonprincipal ultrafilter on $\mathbf{N}$. [If $x \in \mathbf{R}$ is any point not in $\rho(I)\Delta I$ for any rational interval I, then

$$\mathscr{F} = \left\{A \subseteq \mathbf{N} : x \in \rho\left(\bigcup\left\{\left(\frac{-1}{(n+1)}, \frac{-1}{n}\right) \cup \left(\frac{1}{(n+1)}, \frac{1}{n}\right) : n \in A\right\}\right)\right\}$$

is a nonprincipal ultrafilter on $\mathbf{N}$.] Hence the axiom of choice is needed to produce a lifting (see [So]). (See also [C, Chapters 5 and 6].) Shelah showed that it is relatively consistent with ZFC that there are no *Borel* liftings for Lebesgue measure on $\mathbf{R}$ [S]. [Whenever a property of sets is used to describe a lifting, we mean that the sets in the range of the lifting have this property. Thus a Borel lifting is one for which $\rho(E)$ is a Borel set for each measurable set E.] This result has been extended in [J, BJ] (respectively in [BS]) to show that it is relatively consistent with ZFC that there are no projective (resp. Borel) liftings for the Haar measure in any power $2^\kappa = \{0, 1\}^\kappa$ of the two element group.

The aspect of noneffectiveness that will interest us here was first discovered by Johnson in [J]. He showed in ZFC that there can be no translation-invariant Borel lifting for the Haar measure on $\mathbf{R}/\mathbf{Z}$. This result was generalized by Talagrand who proved the following theorem.

0.1. **Theorem** [T]. *For each locally compact abelian group G that is not discrete, there is a Borel set E and a compact set L such that for any translation-invariant lifting ρ, $\rho(E) \cap L$ is not universally measurable and does not have the property of Baire relative to L.*

(The assumption that G is abelian can be dispensed with. This, among other things, was shown by Losert in [L], subsequent to the appearance of Theorem 0.2.)

Kupka and Prikry obtained the following result. (See [KP, Theorem 3.2 and its proof].)

0.2. **Theorem** [KP]. *For each locally compact group G that is not discrete, there is an $(\mathbf{R}/\mathbf{Z})$-valued function f such that for any translation-invariant lifting ρ, $\rho(f)$ does not have the property of Baire.*

The proof of this theorem is fairly elaborate. The function f is not easy to describe. By contrast, in Talagrand's approach, the obstacles, E and L, to having a translation-invariant Borel lifting are easy to describe.

(For example, for $G = \{0, 1\}^\omega$ with the usual Haar measure, we have $L = \{x \in G : x(n) = 0 \text{ for every even } n\}$ and $E = \{x \in G\colon \text{for some even } n, x(n) = 1, \text{ and for the least such } n, |\{j < n : x(j) = 0\}| \text{ is even}\}$.)

The conclusions of Theorems 0.1 and 0.2, however, are somewhat different. On the one hand, it is not clear whether the function f in Theorem 0.2 is universally measurable, nor whether it can be taken to be a characteristic function (and still be independent of ρ). On the other hand, the set L of Theorem 0.1 is typically nowhere dense, so that $\rho(E)$ may very well have the property of Baire. (Indeed, in the example given above for $\{0, 1\}^\omega$, $\rho(E)$ necessarily has the property of Baire since E is open and ρ is strong.)

In this paper we provide (in Theorem 2.1) a generalization of Theorems 0.1 and 0.2, which gives a strong obstacle to the existence of a translation-invariant lifting with nice descriptive properties, using the elegant approach of

[T]. In order to arrange that $\rho(E)$ does not have the property of Baire, some new ideas are needed. Among other things, we will give a version for group quotients (Theorem 1.2) of a well-known theorem of Kuratowski and Ulam on first category subsets of the plane [KU] (or see [O, Theorem 15.1]). It provides, for first category sets, the information that Weil's formula [H, Theorem 63G] provides for sets of measure zero.

1. First category sets and the property of Baire

1.1. **Lemma.** *If X and Y are topological spaces, X is second countable, and $\pi: X \to Y$ is a continuous open surjection, then for every nowhere dense (resp. first category) set $A \subseteq X$, there is a first category set $E \subseteq Y$ such that for all $y \in Y - E$, $A \cap \pi^{-1}(y)$ is nowhere dense (resp. first category) relative to $\pi^{-1}(y)$.*

Proof. It suffices to prove the lemma when A is closed nowhere dense. Suppose $Y_1 = \{y \in Y : \pi^{-1}(y) \cap A$ is somewhere dense in $\pi^{-1}(y)\}$ is second category. For each $y \in Y_1$, choose a basic open set $U(y)$ (from some fixed countable base for the topology of X) such that $\varnothing \neq U(y) \cap \pi^{-1}(y) \subseteq A$. Let $Y_2 \subseteq Y_1$ be a set of second category for which there is an open set U such that $U(y) = U$ for all $y \in Y_2$. But now $Y_2 \subseteq \pi(U) - \pi(U - A)$, which is a contradiction since $\pi(U - A)$ is a dense open subset of $\pi(U)$. □

1.2. **Theorem.** *Let G be a σ-compact locally compact group, $H \subseteq G$ a closed normal subgroup, and $\pi_H: G \to G/H$ the usual projection. Then for every nowhere dense (resp. first category) set $A \subseteq G$, there is a first category set $E \subseteq G/H$ such that for all $y \in (G/H) - E$, $A \cap \pi_H^{-1}(y)$ is nowhere dense (resp. first category) relative to $\pi_H^{-1}(y)$.*

Proof. Again we need only consider the case where A is nowhere dense. Furthermore, since G is completely regular and satisfies the countable chain condition [the Haar measure on G is σ-finite], we may take A to be a closed Baire set. By [H, Theorem 64G], there is a compact normal subgroup K of G such that $\pi_K^{-1}\pi_K(A) = A$ where $\pi_K: G \to G/K$ is the usual projection. Let us denote by $\overline{\pi}_H$ and $\overline{\pi}_K$ the projections $G/H \to G/HK$ and $G/K \to G/HK$, respectively (so that $\overline{\pi}_H\pi_H = \overline{\pi}_K\pi_K$).

By Lemma 1.1, there is a first category F_σ set $E \subseteq G/HK$ such that for each $y \in (G/HK) - E$, $\pi_K(A) \cap \overline{\pi}_K^{-1}(y)$ is nowhere dense in $\overline{\pi}_K^{-1}(y)$. Since $\overline{\pi}_H$ is open, $\overline{\pi}_H^{-1}(E)$ is a first category F_σ in G/H. We claim that for each $x \in (G/H) - \overline{\pi}_H^{-1}(E)$, $A \cap \pi_H^{-1}(x)$ is nowhere dense in $\pi_H^{-1}(x)$.

Suppose not, and let x be a counterexample. Let $y = \overline{\pi}_H(x) \notin E$. Fix $\alpha \in \pi_H^{-1}(x)$ (so $\pi_H^{-1}(x) = \alpha H$). In H, let V be a nonempty open set such that

$$\alpha V \subseteq A \cap \pi_H^{-1}(x) = A \cap \alpha H.$$

Then

$$\pi_K(\alpha V) \subseteq \pi_K(A) \cap \overline{\pi}_K^{-1}(y).$$

We shall have a contradiction if we can show that $\pi_K(\alpha V)$ is open relative to $\overline{\pi}_K^{-1}(y)$.

Note that, since K is compact, VK is open relative to HK. [$VK = V(H \cap K)K = V'K$ where $V' = V(H \cap K)$ is open in H. Now $(H - V')K$ is closed in HK [HR, Theorem 4.4] and HK is the disjoint union of $V'K$

and $(H - V')K$ (since $V' = V'(H \cap K))$.] Thus αVK is open relative to $\alpha HK = \pi_H^{-1}(x)K = \pi_H^{-1}\overline{\pi}_H^{-1}(y) = \pi_K^{-1}\overline{\pi}_K^{-1}(y)$. Thus $\pi_K(\alpha V) = \pi_K(\alpha VK) = \pi_K(\pi_K^{-1}\pi_K(\alpha V))$ is open relative to $\pi_K(\pi_K^{-1}\overline{\pi}_K^{-1}(y)) = \overline{\pi}_K^{-1}(y)$, which is the desired contradiction. □

Kuratowski and Ulam give a counterexample to Theorem 1.2 when G is not σ-compact; see [KU]. To see that the group-theoretic assumptions made on G cannot be removed, consider the following example.

1.3. **Example.** Let X be the Stone space of the measure algebra of $[0, 1]$ and let $Y = [0, 1]$, so that $Y \times X$ is a compact T_2 Radon probability space (with Lebesgue measure on Y, the induced measure on X, and the usual product measure on $Y \times X$) that is self-supporting (= each nonvoid open set has positive measure; the point is that this is a strong chain condition that is also possessed by any σ-compact locally compact group). Let $\pi\colon Y \times X \to Y$ be the projection map. Then there is a first category set $A \subseteq Y \times X$ such that for each $y \in Y$, $A \cap \pi^{-1}(y)$ is a nonvoid open set relative to $\pi^{-1}(y)$.

Construction. Since any perfect Polish space contains a residual copy of the irrationals, we may take for Y any perfect Polish space. Let $\mathscr{K}$ be the set of all compact subsets of $[0, 1]$ with the exponential topology. (See [K, §§17, 21] for the basic properties of this topology.) $\mathscr{K}$ is a compact metrizable space. It is convenient here to use $Y = \{K \in \mathscr{K} : \mu K \geq \frac{1}{2}\}$, which is a closed perfect subspace of $\mathscr{K}$. Let $A \subseteq Y \times X$ be given by

$$A = \bigcup\{\{K\} \times K^* : K \in Y\}$$

where K^* is the set of all ultrafilters in X that contain the equivalence class of K. Then the vertical sections of A are clearly as desired. It remains to see that A is nowhere dense in $Y \times X$.

Let $U \times V$ be a nonvoid basic open set in $Y \times X$. Choose open sets W and $W_i \subseteq W$, $i = 1, 2, \ldots, n$, such that $U = \{K \in Y : K \subseteq W$ and $K \cap W_i \neq \varnothing$ for all $i\}$, and choose a set M of positive measure such that $V = M^*$. Since $U \neq \varnothing$, we have $\mu W > \frac{1}{2}$. Fix any density point of M, and choose a small enough closed interval centered on this density point so that $\mu(W - I) > \frac{1}{2}$ and $W_i - I \neq \varnothing$ for all i. Let

$$U' = \{K \in Y : K \subseteq W - I \text{ and } K \cap (W_i - I) \neq \varnothing \text{ for all } i\},$$

and let $V' = (M \cap I)^*$. Then $U' \times V'$ is a nonvoid open subset of $U \times V$ and $(U' \times V') \cap A = \varnothing$. □

1.4. *Remark.* In the next section we will use only the special case of Theorem 1.2 in which $A = \pi^{-1}(\overline{A})$ for some $\overline{A} \subseteq G/H$. This special case holds in much greater generality, as the next theorem shows. Its proof is very similar to that of Theorem 3(9) in [W] (a result which was brought to our attention by S. Todorcevic) and we leave it to the reader. See [W] for the definition of a weakly α-favorable space. We note here only that locally compact spaces are weakly α-favorable.

1.5. **Theorem.** *If X is weakly α-favorable, $\pi\colon X \to Y$ is an open continuous surjection, and $\overline{A} \subseteq Y$ is second category in Y, then $\pi^{-1}(\overline{A})$ is second category in X.* □

2. Translation-invariant liftings

2.1. **Theorem.** *Let G be a nondiscrete locally compact group with Haar measure μ. Then there is a Borel set $E \subseteq G$ such that for any translation-invariant lifting ρ for (G, μ), $\rho(E)$ is not universally measurable and does not have the property of Baire.*

Proof. As in [KP, first two paragraphs of the proof of Theorem 3.2], we may assume that G is σ-compact, and we may find a compact normal subgroup $K \subseteq G$ such that G/K is second countable and not discrete. Let $\pi\colon G \to G/K$ be the projection map. It will be enough to prove the theorem for G/K. For suppose that we have found a set $E \subseteq G/K$ as in the conclusion of the theorem. Then $\pi^{-1}(E)$ will work for G. The reason for this is as follows. Suppose ρ is a translation-invariant lifting for (G, μ). Then we can define a lifting $\overline{\rho}$ for G/K as follows. For each measurable set $S \subseteq G/K$, $\pi^{-1}(S)$ is invariant under K (by translations both on the left and on the right since K is normal). By the translation-invariance of ρ, $\rho(\pi^{-1}(S))$ is also invariant under K. Let $\overline{\rho}(S) = \rho(\pi^{-1}(S))$. The Haar measure μ on G is carried by π to the Haar measure $\overline{\mu}$ on G/K (because K is compact; see [H, Theorem 63C]). By inner regularity of these measures, it is easy to see that $\overline{\rho}(S)$ is measurable and belongs to the equivalence class of S. It is also easy to see that $\overline{\rho}$ is a Boolean homomorphism and is translation-invariant. Suppose now that $\rho(\pi^{-1}(E))$ has the property of Baire. Let U be the unique regular open set such that $F = U \,\Delta\, \rho(\pi^{-1}(E))$ is first category (see [O, Theorem 4.6]). Since $\pi^{-1}(E)$ is invariant under K, so are $\rho(\pi^{-1}(E))$, U, and F. Thus $\overline{\rho}(E) = \pi(U) \,\Delta\, \pi(F)$. Now $\pi(U)$ is open and $\pi(F)$ is first category by Theorem 1.2. Hence $\overline{\rho}(E)$ has the property of Baire—a contradiction.

To see that $\rho(\pi^{-1}(E))$ is not universally measurable, fix a Radon measure $\overline{\lambda}$ on G/K for which $\overline{\rho}(E)$ is not measurable. Let λ be a Radon measure on G such that $\pi\lambda = \overline{\lambda}$. It is easy to show that if $\rho(\pi^{-1}(E)) = \pi^{-1}(\overline{\rho}(E))$ is λ-measurable, then $\overline{\rho}(E)$ is $\overline{\lambda}$-measurable.

Henceforth we assume that the group G is separable and metrizable. Fix a left-invariant metric on G [HR, Theorem 8.3]. (The word 'diameter' in what follows refers to this nameless metric.)

We shall define open sets $V_r(s)$, where for some $k < \omega$, $r \in 2^k$ and $s \in \omega^k$, such that $V_\varnothing(\varnothing) = G$ and for each $k \neq 0$ and for each $r \in 2^k$ and $s \in \omega^k$, the following conditions are satisfied.

(1) $V_r(s)$ has diameter at most $1/k$.

(2) The sets $V_{r^\frown j}(s^\frown n)$, $j < 2$, $n < \omega$, are pairwise disjoint and their closures are compact subsets of $V_r(s)$.

(3) $V_r(s) - \bigcup_{j<2} \bigcup_{n<\omega} V_{r^\frown j}(s^\frown n)$ is nowhere dense.

(4) $\bigcup_{r\in 2^k, s\in\omega^k} V_r(s)$ has measure at most $1/k$.

We will also define elements $x(s) \in G$ such that the following condition holds. Let us write $\overline{0}$ for the zero element in 2^k (letting the context determine k).

(5) For each nonzero $k < \omega$, and for all $i < k$, $s \in \omega^k$, $r \in 2^k$, if $r \neq \overline{0}$ and $i_1 < \cdots < i_l < k$ are the numbers i such that $r(i) = 1$, then $p(r, s)V_{\overline{0}}(s) = V_r(s)$ where $p(r, s) = x(s|(i_1+1))x(s|(i_2+1))\cdots x(s|(i_l+1))$.

The initial step $k = 1$. Let $\{W_n : n < \omega\}$ list a base for the topology of G. Inductively choose elements $x(n) \in G$ and pairs of open sets

$V_0(n)$, $V_1(n) = x(n)V_0(n)$ with compact closures such that $V_0(n)$ has diameter at most 1 and has measure at most $1/2^{n+2}$, the sets $V_0(n)$, $V_1(n)$ $(n < \omega)$ are pairwise disjoint, and $\bigcup_{k\leq n} V_0(k) \cup V_1(k)$ has nonvoid intersection with W_n but is not dense in G. This is straightforward. Use the fact that G is not discrete and hence perfect, as well as the fact that the points of G have measure zero (G is locally compact, compact sets have finite measure, and G is not discrete). It is clear that all the conditions are now satisfied for $k = 1$.

The inductive step to $k + 1$. Fix a collection $\{\varepsilon(s) : s \in \omega^{k+1}\}$ of positive real numbers such that

$$\sum_{s\in\omega^{k+1}} \varepsilon(s)2^{k+1} \leq \frac{1}{k+1}.$$

For each $s \in \omega^k$, list a base $\{W_n(s) : n < \omega\}$ for the topology of $V_{\bar{0}}(s)$ and inductively choose elements $x(s^\frown n) \in G$ and pairs of open sets $V_{\bar{0}^\frown 0}(s^\frown n)$, $V_{\bar{0}^\frown 1}(s^\frown n) = x(s^\frown n)V_{\bar{0}^\frown 0}(s^\frown n)$ such that $V_{\bar{0}^\frown 0}(s^\frown n)$ has diameter at most $1/(k+1)$ and has measure at most $\varepsilon(s^\frown n)$, the sets $V_{\bar{0}^\frown 0}(s^\frown n)$, $V_{\bar{0}^\frown 1}(s^\frown n)$ $(n < \omega)$ are pairwise disjoint and have compact closures contained in $V_{\bar{0}}(s)$, and $\bigcup_{k\leq n} V_{\bar{0}^\frown 0}(s^\frown k) \cup V_{\bar{0}^\frown 1}(s^\frown k)$ has nonvoid intersection with $W_n(s)$ but is not dense in $V_{\bar{0}}(s)$.

For other $r \in 2^k$, let

$$V_{r^\frown 0}(s^\frown n) = p(r, s_0)V_{\bar{0}^\frown 0}(s^\frown n) = p(r^\frown 0, s^\frown n)V_{\bar{0}^\frown 0}(s^\frown n),$$
$$V_{r^\frown 1}(s^\frown n) = p(r, s)x(s^\frown n)V_{\bar{0}^\frown 0}(s^\frown n) = p(r^\frown 1, s^\frown n)V_{\bar{0}^\frown 0}(s^\frown n).$$

Since translation by $p(r, s)$ is a measure-preserving isometry between $V_{\bar{0}}(s)$ and $V_r(s)$, conditions (1), (2), and (3) are satisfied. Condition (4) holds by the choice of the $\varepsilon(s)$'s, and condition (5) holds by definition of the $V_r(s)$'s.

Let $U_k = \bigcup\{V_r(s) : r \in 2^k,\ s \in \omega^k\}$. This is a dense open set of measure at most $1/k$. Thus $\bigcap_k U_k$ is a dense G_δ set of measure zero.

2.2. *Remark.* By condition (1), the sets $V_r(s)$ that contain a given point $x \in \bigcap_k U_k$ form a base at x.

Now define $E \subseteq G$ as follows.

For each $k \neq 0$ and for each $s \in \omega^k$, partition $V_{\bar{0}}(s) - U_{k+1}$ into two disjoint sets of equal measure, $A_{\bar{0}}(s)$ and $B_{\bar{0}}(s)$. For $r \in 2^k$, let $l(r) = |\{i < k : r(i) = 1\}|$. Define

$$A_r(s) = \begin{cases} p(r, s)A_{\bar{0}}(s), & l(r) \text{ even}, \\ p(r, s)B_{\bar{0}}(s), & l(r) \text{ odd}, \end{cases}$$

so that $\{A_r(s), B_r(s)\}$ is a partition of $V_r(s) - U_{k+1}$ into two sets of equal measure. (Note that $p(r, s)(V_{\bar{0}}(s) - U_{k+1}) = V_r(s) - U_{k+1}$.)

Let $E = \bigcup_{r,s} A_r(s)$.

Suppose that for some translation-invariant lifting ρ, $\rho(E)$ has the property of Baire. Then one of $\rho(E)$, $\rho(E^c)$ contains a nonvoid open set modulo a first category set, say the former.

Then $\rho(E) \cap \bigcap_k U_k$ contains a nonvoid open set modulo a first category set, and by Remark 2.2, there is a $k \neq 0$ and there are sequences $r \in 2^k$ and $s \in \omega^k$ such that $V_r(\bar{s}) \subseteq \rho(E)$ modulo a first category set.

Let $s = \bar{s}^\frown n$ be any extension of $\bar{s}$.

Choose $y \in (V_{r^\frown 0}(s) \cap \rho(E))$ such that $p(r^\frown 1, s)p(r^\frown 0, s)^{-1}y \in V_{r^\frown 1}(s) \cap \rho(E)$. For each $i \geq k+1$, we have

$$p(r^\frown 1, s)p(r^\frown 0, s)^{-1}(E \cap V_{r^\frown 0}(s) \cap (U_i - U_{i+1})) \\ = [V_{r^\frown 1}(s) - (E \cap V_{r^\frown 1}(s))] \cap (U_i - U_{i+1}).$$

[For each $a \in 2^i$, $b \in \omega^i$ such that $a \supseteq r^\frown 0$, $b \supseteq s$, let $\overline{a}$ denote the element of 2^k that agrees with a everywhere except at k (and $\overline{a}(k) = 1$). We have $p(\overline{a}, b)p(a, b)^{-1} = p(r^\frown 1, s)p(r^\frown 0, s)^{-1}$. Also one of $l(a)$, $l(\overline{a})$ is even, and the other one is odd. Hence

$$p(r^\frown 1, s)p(r^\frown 0, s)^{-1}(E \cap V_a(b) - U_{i+1}) = [V_{\overline{a}}(b) - (E \cap V_{\overline{a}}(b))] - U_{i+1}.$$

Taking the union over all such a, b gives the desired equality.]

Now taking the union over all i we get

$$p(r^\frown 1, s)p(r^\frown 0, s)^{-1}(E \cap V_{r^\frown 0}(s)) = V_{r^\frown 1}(s) - (E \cap V_{r^\frown 1}(s)) - \bigcap_k U_k.$$

Applying ρ to this equality we see that $\rho(V_{r^\frown 1}(s))$ is the disjoint union of

$$p(r^\frown 1, s)p(r^\frown 0, s)^{-1}\rho(E \cap V_{r^\frown 0}(s)) \quad \text{and} \quad \rho(E \cap V_{r^\frown 1}(s)).$$

However, this is impossible since the choice of y guarantees that

$$p(r^\frown 1, s)p(r^\frown 0, s)^{-1}y$$

belongs to both of these sets. (We have $p(r^\frown 1, s)p(r^\frown 0, s)^{-1}y \in \rho(V_{r^\frown 1}(s))$ since ρ is a strong lifting.)

To see that $\rho(E)$ is not universally measurable, define $h: 2^\omega \to G$ by $h(f) =$ the unique member of $\bigcap_k V_{f|k}(\overline{0})$. Conditions (1) and (2) ensure that h is a homeomorphism onto its range. If we let λ be the image under h of the usual Haar measure on 2^ω, then as in [T], $\rho(E)$ is not λ-measurable. □

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