NORMAL SPACES WHOSE STONE-ČECH REMAINDERS
HAVE COUNTABLE TIGHTNESS

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Abstract. We prove, assuming PFA, that each normal space whose Stone-
Čech remainder has countable tightness is ACRIN. A normal space $X$ is called
ACRIN if each of its regular images is normal. Fleissner and Levy proved that if
$X$ is normal and every countably compact subset of the Stone-Čech remainder
$\beta X \setminus X$ is closed in $\beta X \setminus X$, then $X$ is ACRIN. They asked if each normal
space whose Stone-Čech remainder has countable tightness is ACRIN. Theorem
2 gives the positive answer assuming the Proper Forcing Axiom.

It is well known that the tightness of $\beta \omega \setminus \omega$ is $2^\omega$. Since every not countably
compact Hausdorff space contains a closed copy of $\omega$, the next lemma is easy
to prove.

Lemma 1. If $X$ is a normal space and $\beta X \setminus X$ has countable tightness, then $X$
is countably compact.

Theorem 2 (PFA). If $X$ is a normal space and $\beta X \setminus X$ has countable tightness,
then $X$ is ACRIN.

Proof. Let $f: X \to Y$ be a continuous map and $Y$ be regular. We prove that
$Y$ is normal. By virtue of Lemma 5 of [FL] there exist $Z$ and $bf$ such that
$X \subseteq Z \subseteq \beta X$ and $bf$ is a perfect map from $Z$ onto $Y$ with $bf|_X = f$.
Since $\beta X \setminus X$ has countable tightness, it is easy to see that the spaces $X$, $Y$,
and $Z$ are all countably compact. Since perfect mappings preserve normality,
we only need to prove that $Z$ is normal. Let $K$ and $L$ be two disjoint closed
subsets of $Z$. We will prove that $K^\beta X \cap L^\beta X = \emptyset$. Since $X$ is normal, we
have $K \cap X^\beta X \cap L \cap X^\beta X = \emptyset$. Take open subsets $U$ and $V$ of $\beta X$
such that $U^\beta X \cap V^\beta X = \emptyset$, $K \cap X^\beta X \subseteq U$, and $L \cap X^\beta X \subseteq V$. Let $U' = U \setminus L^\beta X$,
$V' = V \setminus K^\beta X$, and $T = (K \cup L^\beta X) \setminus (U' \cup V')$. Obviously $T$ is a closed subset
of $\beta X$ and is contained in $\beta X \setminus X$. Thus $T$ is a compact space of countable
tightness. Furthermore, $K \setminus U$ and $L \setminus V$ are contained in $T$ and are closed
subsets of the countably compact space $Z$. We have proved that $K \setminus U$ and
$L \setminus V$ are countably compact subsets of a compact space of countable tightness.
By virtue of Balogh’s Theorem [Ba, 2.1], $K \setminus U$ and $L \setminus V$ are compact. Thus

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we have
\[
\overline{K}^\beta X \cap \overline{L}^\beta X = (\overline{K \cap U}^\beta X \cup (K \setminus U)) \cap (\overline{L \cap V}^\beta X \cup (L \setminus V)) \\
= (\overline{K \cap U}^\beta X \cap (L \setminus V)) \cup (\overline{L \cap V}^\beta X \cap (K \setminus U)) \\
\subseteq (\overline{K}^\beta X \cap L) \cup (\overline{L}^\beta X \cap K) = \emptyset .
\]

We are done.

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REFERENCES
