FEKETE-SZEGÖ INEQUALITIES
FOR CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let \( K(\beta) \) denote the class of normalised close-to-convex functions of order \( \beta \) defined in the unit disc, and let \( f \in K(\beta) \) with \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \). Sharp bounds are obtained for \( |a_3 - \mu a_2^2| \), where \( \mu \) is real.

1. Introduction

Let the function \( f \) be given by

\[
f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (|z| < 1).
\]

A classical result of Fekete and Szegö [1] determines the maximum value of \( |a_3 - \mu a_2^2| \), as a function of the real parameter \( \mu \), for the class of univalent functions \( f \). There are now several results of this type in the literature, each of them dealing with \( |a_3 - \mu a_2^2| \) for various classes of functions \( f \) (see, e.g., [3-6]).

In this paper we consider the problem for the class \( K(\beta) \) of close-to-convex functions of order \( \beta \) in the sense of Pommerenke [7]. Thus \( f \in K(\beta) \) if and only if \( f \) is given by (1) and for some starlike function \( g \) satisfies

\[
\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi \beta}{2}
\]

for \( |z| < 1 \) and \( \beta \geq 0 \). A recent paper by Abdel-Gawad and Thomas [2] contains a partial proof of the following theorem, the last two inequalities remaining unproved for \( \beta > 1 \).

Theorem. Let \( f \in K(\beta) \) and be given by (1). Then for \( \beta \geq 0 \)

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{3}(1 + (2 - 3\mu)(\beta + 1)^2) & \text{if } \mu \leq \frac{2\beta}{3(\beta+1)}, \\
\frac{1}{3}\left(1 + 2\beta + \frac{2(2-3\mu)}{2-\beta(2-3\mu)}\right) & \text{if } \frac{2\beta}{3(\beta+1)} \leq \mu \leq \frac{2}{3}, \\
\frac{1}{3}(1 + 2\beta) & \text{if } \frac{2}{3} \leq \mu \leq \frac{2(\beta+2)}{3(\beta+1)}, \\
\frac{1}{3}(-1 + (3\mu - 2)(\beta + 1)^2) & \text{if } \mu \geq \frac{2(\beta+2)}{3(\beta+1)}. 
\end{cases}
\]

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For each $\mu$ there are functions in $K(\beta)$ such that equality holds in all cases.

We give a simple proof of the complete theorem.

2. Proof of the theorem

We shall require the following:

Lemma [8, pp. 41, 166]. Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ satisfy $\text{Re} h(z) > 0 (|z| < 1)$. Then $|c_k| \leq 2$ ($k \geq 1$) and

$$|c_2 - \frac{1}{2} c_1^2| \leq 2 - \frac{1}{2} |c_1|^2.$$ 

Let $f \in K(\beta)$. Then it follows from (2) that we may write

$$zf'(z) = g(z)h^p(z),$$

where $g$ is starlike and $h$ has positive real part. Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, and let $h$ be given as in the lemma above. Then by equating coefficients we obtain

$$2a_2 = \beta c_1 + b_2$$

and

$$3a_3 = \frac{1}{2} \beta (\beta - 1)c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3,$$

so, with $X = \frac{1}{2}(2 - 3\mu)$,

$$3(a_3 - \mu a_2^2) = b_3 - \frac{3}{4} \mu b_2^2 + \beta (c_2 + \frac{1}{2}(\beta X - 1)c_1^2) + \beta X c_1 b_2. \quad (3)$$

Since rotations of $f$ also belong to $K(\beta)$, we may assume, without loss of generality, that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\text{Re}(a_3 - \mu a_2^2)$.

For some function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ ($|z| < 1$) with positive real part, we have $z g''(z) = g(z)p(z)$; hence, by equating coefficients, $b_2 = p_1$ and $b_3 = \frac{1}{2}(p_2 + p_1^2)$. So by the lemma,

$$\text{Re}(b_2 - \frac{3}{4} \mu b_2^2) = \frac{1}{2} \text{Re}(p_2 - \frac{1}{2} p_1^2) + \frac{3}{4}(1 - \mu) \text{Re} p_1^2 \leq 1 - p^2 + (1 + 2X)p^2 \cos 2\phi, \quad (4)$$

where $b_2 = p_1 = 2 \rho e^{i\phi}$ for some $\rho$ in $[0, 1]$. We also have

$$\text{Re}(c_2 + \frac{1}{2}(\beta X - 1)c_1^2) = \text{Re}(c_2 - \frac{1}{2} c_1^2) + \frac{1}{2} \beta X \text{Re} c_1^2 \leq 2(1 - r^2) + 2\beta X r^2 \cos 2\theta, \quad (5)$$

where $c_1 = 2r e^{i\theta}$ for some $r$ in $[0, 1]$. From (3)–(5) we obtain

$$\text{Re } 3(a_3 - \mu a_2^2) \leq 1 - p^2 + (1 + 2X)p^2 \cos 2\phi + 2\beta(1 - r^2 + r^2 \beta X \cos 2\theta) + 4\beta X r \rho \cos(\theta + \phi), \quad (6)$$

and we now proceed to maximize the right-hand side of (6). This function will be denoted $\varphi(X)$ whenever all the parameters except $X$ are held constant.

Assume that $2\beta/3(\beta + 1) \leq \mu \leq 2/3$ so that $0 \leq X \leq 1(1 + \beta)$. The expression $-t^2 + t^2 \beta X \cos 2\theta + 2Xt$ is largest when $t = X/(1 - \beta X \cos 2\theta)$, so

$$-t^2 + t^2 \beta X \cos 2\theta + 2Xt \leq \frac{X^2}{1 - \beta X \cos 2\theta} \leq \frac{X^2}{1 - \beta X}.$$
Thus

$$\psi(X) \leq 1 + 2X + 2\beta \left(1 + \frac{X^2}{1 - \beta X}\right) = 1 + 2\beta + \frac{2(2 - 3\mu)}{2 - \beta(2 - 3\mu)},$$

and with (6) this establishes the second inequality in the theorem.

It is now to prove the first inequality. Let $\mu < 2\beta/3(\beta + 1)$, so that $X > 1/(1 + \beta)$. With $X_0 = 1/(1 + \beta)$ we have

$$\psi(X) = \psi(X_0) + 2(X - X_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\rho \beta r \cos(\theta + \phi))$$

$$\leq \psi(X_0) + 2(X - X_0)(\beta + 1)^2 \leq 1 + (2 - 3\mu)(\beta + 1)^2$$

as required.

Let $X_1 = -1/(1 + \beta)$. We shall find that $\psi(X_1) \leq 2\beta + 1$, and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\psi(X) \leq \psi(X_1) + 2|X - X_1|(\beta + 1)^2 \leq -1 + (3\mu - 2)(\beta + 1)^2$$

if $X \leq X_1$, that is, $\mu \geq 2(\beta + 2)/3(\beta + 1)$. Also, for $0 \leq \lambda \leq 1$,

$$\psi(\lambda X_1) = \lambda \psi(X_1) + (1 - \lambda)\psi(0)$$

$$\leq \lambda(2\beta + 1) + (1 - \lambda)(2\beta + 1) = 2\beta + 1,$$

so $\psi(X) \leq 2\beta + 1$ for $X_1 \leq X \leq 0$, i.e., $2/3 \leq \mu \leq 2(\beta + 2)/3(\beta + 1)$.

We now show that $\psi(X_1) \leq 2\beta + 1$. We have

$$-t^2 + t^2 \beta X \cos 2\theta + 2X t \rho \cos(\theta + \phi) \leq \frac{X^2 \rho^2 \cos 2(\theta + \phi)}{1 - \beta X \cos 2\theta}$$

for all real $t$, so

$$\psi(X) - 1 - 2\beta \leq \rho^2 \left[-1 + (1 + 2X) \cos 2\phi + \frac{\beta X^2(1 + \cos 2(\theta + \phi))}{1 - \beta X \cos 2\theta}\right].$$

Thus we consider the inequality

$$\beta X^2(1 + \cos 2(\theta + \phi)) + (1 - \beta X \cos 2\theta)(-1 + (1 + 2X) \cos 2\phi) \leq 0$$

with $X = X_1$. After some simplification this becomes

$$\beta^2(\cos 2\phi - 1)(\cos 2\theta + 1) - \beta(1 + \cos 2\theta + \sin 2\theta \sin 2\phi) - 1 - \cos 2\phi \leq 0,$$

which is true if

$$(7) \quad 2\beta^2 \sin^2 \phi \cos^2 \theta + 2\beta \cos \theta \sin \theta \cos \phi \sin \phi + \cos^2 \phi \geq 0.$$

Now, for all real $t$,

$$2t^2 + 2 \sin \theta \cos \phi t + \cos^2 \phi \geq 0,$$

so by taking $t = \beta \sin \phi \cos \theta$, we obtain (7). This completes the proof of the inequalities.

An examination of the proof shows that the four inequalities in the theorem are sharp if we take $c_1 = c_2 = b_2 = 2$, $b_3 = 3$ in the first case; $c_1 = 2(2 - 3\mu)/2 - \beta(2 - 3\mu)$, $c_2 = b_2 = 2$, $b_3 = 3$ in the second; $c_1 = b_2 = 0$,
c_2 = 2, \ b_3 = 1 \ in \ the \ third; \ and \ \ c_1 = b_2 = 2i, \ c_2 = -2, \ b_3 = -3 \ in \ the \ last.

The corresponding functions f may be defined by taking f'(z) respectively as

\[
\frac{1}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^\beta, \quad \frac{1}{(1 - z)^2} \left( \frac{1 + z}{1 - z} + (1 - \lambda) \frac{1 - z}{1 + z} \right)^\beta, \\
\frac{1}{1 - z^2} \left( \frac{1 + z^2}{1 - z^2} \right)^\beta, \quad \frac{1}{(1 - iz)^2} \left( \frac{1 + iz}{1 - iz} \right)^\beta,
\]

where, in the second case,

\[
\lambda = \frac{2 + (1 - \beta)(2 - 3\mu)}{2(2 - \beta(2 - 3\mu))}.
\]

REFERENCES


