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BERNSTEIN INEQUALITIES FOR A CLASS OF RANDOM VARIABLES

M. SCHMUCKENSCHLAEGER

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ABSTRACT. We prove a deviation inequality for sums of i.i.d. random variables.

Under the condition of the existence of an exponential moment, the Bernstein inequalities give an upper bound for the measure of the deviation of the arithmetic mean for independent identically distributed random variables with expectation zero. (For a precise formulation see [H] or [P]; for an application see, for example, [BLM].) We consider the weaker assumption

$$\mathbf{E}\exp(X^\alpha) < \infty \quad \text{for some } \alpha \in (0, 1)$$

and a nonnegative random variable X and show for sufficiently large t, i.e., for $t > t_0$, where t_0 only depends on α and $\mathbf{E}X$, that

$$\mathbf{P}\left(\frac{1}{n}\sum_{j=1}^{n} X_j - \mathbf{E}X > t\mathbf{E}X\right) \leq \exp-(c(\alpha)(nt\mathbf{E}X)^\alpha).$$

The method of proof follows a paper by Schechtman and Zinn [SZ]. The following lemma is elementary.

Lemma. (i) *If* $j \leq N := [\frac{n+1}{2}]$ *then* $\log\binom{n}{j} \leq 2j\log(\frac{n}{j})$.

(ii) $\sum_{j=1}^{\infty} j^{-1/\alpha} \leq (1-\alpha)^{-1}\ \forall\, 0 < \alpha < 1$.

(iii) $\sum_{j=1}^{N}(\log(\frac{nc}{j}))^{1/\alpha} \leq nc\Gamma(1+\frac{1}{\alpha}),\ c \geq 1$.

Proof. (i) is an immediate consequence of Stirling's formula and the fact that the function

$$x \mapsto x\log\left(\frac{a}{x}\right) - (a-x)\log\left(\frac{a}{a-x}\right)$$

is nondecreasing on $(\frac{a}{2}, a)$ and vanishes at $\frac{a}{2}$. For $j = [\frac{n+1}{2}]$ the inequality can be checked directly.

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(iii) is proved by replacing the sum by an integral:

$$\begin{aligned}\sum_{j=1}^{N} \log^{1/\alpha}\left(\frac{cn}{j}\right) &\le \log^{1/\alpha}(cn) + \int_1^N \log^{1/\alpha}\left(\frac{cn}{t}\right) dt \\ &= \log^{1/\alpha}(cn) + cn \int_{\log(cn/N)}^{\log(cn)} x^{1/\alpha} \exp(-x)\, dx \\ &< \log^{1/\alpha}(cn) + cn\Gamma\left(1+\frac{1}{\alpha}\right) - cn \int_{\log(cn)}^{\infty} x^{1/\alpha} \exp(-x)\, dx \\ &\le \log^{1/\alpha}(cn) + cn\Gamma\left(1+\frac{1}{\alpha}\right) - cn\frac{1}{cn}\log^{1/\alpha}(cn) \\ &= cn\Gamma\left(1+\frac{1}{\alpha}\right). \quad \square\end{aligned}$$

Let $0 < \alpha < 1$, and let X be a nonnegative random variable such that

$$\mathbf{E}\exp(X^\alpha) = A < \infty.$$

Let $(X_j)_{j=1}^n$ be independent copies of X, and let (X_j^*) be the nonincreasing rearrangement of (X_j). Then for all $t > 0$,

$$\begin{aligned}\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j > t\right) &= \mathbf{P}\left(\sum_{j\le n} X_j^* > nt\right) \\ &\le \mathbf{P}\left(\sum_{j\le N} X_j^* > \frac{1}{2}nt\right) + \mathbf{P}\left(\sum_{j>N} X_j^* > \frac{1}{2}nt\right) \\ &\le 2\mathbf{P}\left(\sum_{j\le N} X_j^* > \frac{1}{2}(1+t)nE\right)\end{aligned}$$

where $N = [(n+1)/2]$. Choose $p_j > 0$ in such a way that $\sum_{j\le N} p_j \le 1$. Then

$$\mathbf{P}\left(\sum_{j\le N} X_j^* > \frac{1}{2}nt\right) \le \sum_{j\le N} \mathbf{P}\left(X_j^* > \frac{1}{2}p_j nt\right).$$

Since for all $s > 0$

$$[X_j^* > s] = \bigcup_{A(j)} \left\{ \bigcap_{k\in A(j)} [X_k > s] : A(j) \subseteq \{1, \dots, n\}, |A(j)| = j \right\}$$

and

$$\mathbf{P}(X > s) \le A\exp(-s^\alpha),$$

by the lemma above we get

$$\begin{aligned}\mathbf{P}(X_j^* > s) &\le A^j \binom{n}{j} \exp(-js^\alpha) \\ &\le \exp\left(j\left(\log A + 2\log\left(\frac{n}{j}\right) - s^\alpha\right)\right),\end{aligned}$$

$$\mathbf{P}\left(\sum_{j\le N} X_j^* > \frac{1}{2}nt\right) \le \sum_{j\le N} \exp\left(j\left(\log A + 2\log\left(\frac{n}{j}\right) - \left(\frac{1}{2}p_j nt\right)^\alpha\right)\right).$$

We now choose the p_j so that we have the inequality

$$j\left(\log A + 2\log\left(\frac{n}{j}\right) - \left(\frac{p_j}{2}nt\right)^\alpha\right) \le -(cnt)^\alpha$$

for some constant c, i.e.,

$$\begin{aligned} p_j &= \left((2c)^\alpha \frac{1}{j} + \left(\frac{2}{nt}\right)^\alpha \left(\log A + 2\log\left(\frac{n}{j}\right)\right)\right)^{1/\alpha} \\ &\le 2^{1/\alpha-1}\left((2c)\frac{1}{j^{1/\alpha}} + \frac{2}{nt}\left(\log A + 2\log\left(\frac{n}{j}\right)\right)^{1/\alpha}\right) \\ &= 2^{1/\alpha} c j^{-1/\alpha} + \frac{2^{2/\alpha}}{nt}\log^{1/\alpha}\left(\frac{n\sqrt{A}}{j}\right). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{j=1}^N p_j &\le 2^{1/\alpha} c \sum_{j=1}^N j^{-1/\alpha} + \frac{2^{2/\alpha}}{nt}\sum_{j=1}^N \log^{1/\alpha}\left(\frac{n\sqrt{A}}{j}\right) \\ &\le 2^{1/\alpha} c \frac{1}{1-\alpha} + \frac{2^{2/\alpha}}{nt} n\sqrt{A}\,\Gamma\left(1+\frac{1}{\alpha}\right). \end{aligned}$$

For $c = (1-\alpha)/(2^{1/\alpha+1})$ and $t \ge \sqrt{A}4^{1/\alpha}2\Gamma(1+1/\alpha)$ we therefore have $\sum_{j=1}^N p_j \le 1$. For t satisfying the above inequality we obtain

$$\begin{aligned} \mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j > t\right) &\le 2\sum_{j\le N} \exp\left(-c^\alpha n^\alpha t^\alpha\right) \le 2N\exp -(cnt)^\alpha \\ &= \exp(\log 2N - (cnt)^\alpha). \end{aligned}$$

Since

$$\begin{aligned} (ct)^\alpha &\ge \left(\frac{1-\alpha}{2^{1/\alpha+1}}\sqrt{A}2^{2/\alpha+1}\Gamma\left(1+\frac{1}{\alpha}\right)\right)^\alpha \\ &= 2\left((1-\alpha)\Gamma\left(1+\frac{1}{\alpha}\right)\right)^\alpha A^{\alpha/2} =: 2D(\alpha), \end{aligned}$$

we get for values of n with $\log(n+1) \le (\frac{c}{2}nt)^\alpha$ (in particular, whenever $n \ge 4(\alpha D(\alpha))^{-2/\alpha}$) the estimate

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j > t\right) \le \exp -\left(\frac{c}{2}nt\right)^\alpha.$$

Theorem. *Let $0 < \alpha < 1$, and let X be a nonnegative random variable such that $\mathbf{E}\exp(X^\alpha) = A < \infty$. If $(X_j)_{j=1}^n$ are independent copies of X, then for*

$$t \ge \sqrt{A}\,2^{2/\alpha+1}\Gamma\left(1+\frac{1}{\alpha}\right) \quad \textit{and} \quad n \ge \frac{4}{A}\left(\alpha^{1/\alpha}(1-\alpha)\Gamma\left(1+\frac{1}{\alpha}\right)\right)^{-2}$$

we have

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j > t\right) \le \exp - \left(\frac{1-\alpha}{2^{1/\alpha+2}} nt\right)^{\alpha}.$$

In terms of the Orlicz-norm N_α associated with the function $\exp(t^\alpha) - 1$, i.e.,

$$N_\alpha(X) := \inf\left\{ l > 0 : E\exp\left(\frac{|X|}{l}\right)^{\alpha} \le 2\right\},$$

we get the following

Corollary. *Let* $0 < \alpha < 1$, *and let* X *be a nonnegative random variable such that* $N_\alpha(X) < \infty$. *Then for*

$$t \ge 2^{2/\alpha+3/2}\Gamma\left(1+\frac{1}{\alpha}\right) N_\alpha(X) \quad \textit{and} \quad n \ge 2\left(\alpha^{1/\alpha}(1-\alpha)\Gamma\left(1+\frac{1}{\alpha}\right)\right)^{-2}$$

we have

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j > t\right) \le \exp - \left(\frac{1-\alpha}{2^{1/\alpha+2}} n \frac{t}{N_\alpha(X)}\right)^{\alpha}.$$

The result is optimal, as can be shown by choosing for X a random variable with density function

$$c_\alpha \exp(-t^\alpha), \qquad c_\alpha = \Gamma\left(1+\frac{1}{\alpha}\right)^{-1}.$$

In this case we conclude that

$$\begin{aligned}
\mathbf{P}(X > x) &= \frac{c_\alpha}{\alpha x^{\alpha-1}} \int_x^\infty \alpha x^{\alpha-1} \exp - t^\alpha \, dt \\
&\ge \frac{c_\alpha}{\alpha x^{\alpha-1}} \int_x^\infty \alpha t^{\alpha-1} \exp - t^\alpha \, dt = \frac{c_\alpha}{\alpha} x^{1-\alpha} \exp(-x^\alpha) \\
&\ge \frac{c_\alpha}{\alpha} \exp(-x^\alpha) \quad \text{for } x \ge 1.
\end{aligned}$$

Thus

$$\mathbf{P}\left(\sum^n X_i > nt\right) \ge P(X > nt) \ge \frac{c_\alpha}{\alpha} \exp - (nt)^\alpha.$$

Remark. It is easy to get an estimate for

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j < tE\right) \qquad (t \le 1,\ E := \mathbf{E}X).$$

For $M > 0$ we clearly have for $X_j' = X_j I_{[X_j \le M]}$

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\le n} X_j < tE\right) \le \mathbf{P}\left(\frac{1}{n}\sum X_j' < tE\right).$$

Since $||X_j'||_\infty \leq M$, the Bernstein inequality yields for $\mathbf{E}X' - tE \geq 0$ and some constant c_1

$$\mathbf{P}\left(\frac{1}{n}\sum_{j\leq n} X_j < tE\right) \leq \exp - \left(c_1 n \left(\frac{\mathbf{E}X' - tE}{M}\right)^2\right).$$

Since

$$\mathbf{E}XI_{[X>M]} \leq \mathbf{E}X^2 M^{-1} = \frac{1}{M}\mathbf{E}X^2,$$

we get for $M = 2||X||_2^2||X||_1^{-1} \leq c'(\alpha)N_\alpha(X)^2E^{-1}$ and $t \leq 1/2$,

$$P\left(\frac{1}{n}\sum X_j < tE\right) \leq \exp\left(-c(\alpha)n\left(\frac{E}{N_\alpha(x)}\right)^4\left(\frac{1}{2} - t\right)^2\right).$$

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Institut für Mathematik, J. Kepler Universität, Linz, Austria