REPRESENTATIONS OF \( \text{AlgLat}(T) \)

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Dedicated to George Maltese on his 60th birthday

Abstract. For a hyponormal operator \( T \) with the property that the boundary of the essential spectrum is of planar Lebesgue measure zero, it is proved that the operator algebra \( \text{AlgLat}(T) \) generated by the invariant subspace lattice of \( T \) is commutative. If in addition \( T \) is a pure hyponormal operator, then \( \text{AlgLat}(T) \) is shown to be contained in the bicommutant of \( T \). These are particular cases of more general results obtained for restrictions and quotients of operators decomposable in the sense of Foiaș.

An operator \( T \in L(H) \) on a complex Hilbert space is called reflexive if the operator algebra \( \text{AlgLat}(T) \) generated by the invariant subspace lattice of \( T \) is as small as it can be, namely, coincides with the closure of the algebra of all polynomials in \( T \) with respect to the weak operator topology. In [16] Sarason proved that normal operators and analytic Toeplitz operators are reflexive. In [7] Deddens was able to show that all isometries are reflexive. Using the Scott Brown technique Olin and Thomson [15] proved that, more general, all subnormal operators are reflexive. In 1987 Scott Brown [3] applied his methods to prove invariant subspace results for hyponormal operators. In [4] Chevreau, Exner, and Pearcy formulated the conjecture that all hyponormal operators are reflexive.

As a modest step in this direction we shall show that for each hyponormal operator \( T \in L(H) \), for which the boundary of the essential spectrum has planar Lebesgue measure zero, the algebra \( \text{AlgLat}(T) \) is commutative. If, in addition, \( T \) is pure, i.e., has no nontrivial normal reducing parts, then \( \text{AlgLat}(T) \) is shown to be contained in the bicommutant of \( T \).

1. Fredholm theory

Let us denote by \( Y \) and \( Z \) complex Banach spaces that are dual to each other in the sense that either \( Z = Y' \) or \( Y = Z' \). We fix continuous linear operators \( A \in L(Y), \ B \in L(Z) \) with

\[ \langle Ay, z \rangle = \langle y, Bz \rangle \quad (y \in Y, \ z \in Z). \]
Our aim is to represent the operator algebra generated by the invariant subspace lattice of $A$ as an algebra of bounded analytic functions on a suitable open subset of the complex plane.

As usual we denote by $\sigma_e(A)$ the essential spectrum of $A$, i.e., the set of all complex numbers $\lambda$ such that $\text{Ker}(\lambda - A)$ or $Y/\text{Im}(\lambda - A)$ is infinite dimensional. Moreover, we write $\text{Lat}(A)$ for the lattice of all closed invariant subspaces of $A$. Let us consider the operator algebra $\sigma(\text{AlgLat}(A))$ consisting of all $\sigma(Y, Z)$-continuous linear operators in $L(Y)$ that leave invariant all $\sigma(Y, Z)$-closed spaces in $\text{Lat}(A)$. If $H$ is a hole in $\sigma_e(A)$, i.e., a bounded component of $\partial(A) - \partial^1(A)$, then we write $\text{ind}_H(A)$ for the constant value of the index $\text{ind}(\lambda - A)$ on $H$. There is an elementary way to represent $\sigma(\text{AlgLat}(A))$ as an algebra of bounded analytic functions on the holes $H$ of $\sigma_e(A)$, which are contained in the spectrum $\sigma(A)$ of $A$.

Let us fix an operator $C \in \sigma(\text{AlgLat}(A))$ and a hole $H \subset \sigma(A)$ of $\sigma_e(A)$. If $\text{ind}_H(A) \leq 0$, then by a result of Finch [9, Theorem 9] for each $\lambda_0 \in H$ there is an open neighbourhood $U$ of $\lambda_0$ and an analytic function $f \in O(U, Z)$ without zeros on $U$ such that $(\lambda - B)f(\lambda) = 0$ ($\lambda \in U$). If $\text{ind}_H(A) > 0$, then everything remains true with $Z$ and $B$ replaced by $Y$ and $A$. Since $C' \in \sigma(\text{AlgLat}(B))$, in the first case for each $\lambda \in H$ there is a unique complex number denoted by $h(\lambda)$ with

$$C|_{\text{Ker}(\lambda - B)} = h(\lambda)I_{\text{Ker}(\lambda - B)}.$$

In the second case $h(\lambda)$ is defined by the relation

$$C|_{\text{Ker}(\lambda - A)} = h(\lambda)I_{\text{Ker}(\lambda - A)}.$$

**Lemma 1.1.** In both cases we obtain a norm continuous algebra homomorphism

$$\Phi_H : \sigma(\text{AlgLat}(A)) \rightarrow H^\infty(H), \quad C \mapsto h$$

with $\Phi(p(A)) = p$ for all complex polynomials $p$.

**Proof.** We only show why for a given operator $C$ the function $h$ is analytic. If $\text{ind}_H(A) \leq 0$, then for $\lambda_0 \in H$ we choose an analytic function $f \in O(U, Z)$ on a neighbourhood of $\lambda_0$ as explained above. The analyticity of $h$ on $U$ follows from the observation that for $\lambda \in U$ and $y \in Y$

$$\langle y, C'f(\lambda) \rangle = h(\lambda)\langle y, f(\lambda) \rangle.$$

The argument in the case $\text{ind}_H(A) > 0$ is analogous.

We used the eigenspaces of $B$, respectively $A$, to define the representing function $h$ for a given operator $C$. The next result describes what happens if the eigenspaces are replaced by generalized eigenspaces. To formulate it we denote by $\Sigma_H$ the set of discontinuity points of $\dim \text{Ker}(\lambda - A)$ as a function of $\lambda$ on $H$. It is well known that $\Sigma_H$ is a discrete subset of $H$ [11, Satz 104.4].

**Proposition 1.2.** If $\text{ind}_H(A) \leq 0$, then for $\lambda \in H \setminus \Sigma_H$ we have

$$C'|_{\text{Ker}(\lambda - B)^n} = \sum_{j=0}^{n-1} \frac{(h^{(j)}(\lambda)/j!)(B - \lambda)^j}{\text{Ker}(\lambda - B)^n}$$

for all integers $n \geq 1$. In the case $\text{ind}_H(A) > 0$ this formula remains valid if $C'$ and $B$ are replaced by $C$ and $A$.
Proof. We only consider the case \( \text{ind}(A) \leq 0 \). Let \( \lambda_0 \in H \setminus \Sigma_H \) be fixed. By induction one can show that for each integer \( n \geq 1 \) and each \( z \in \text{Ker}(\lambda_0 - B)^n \) there are analytic functions \( f_1, \ldots, f_n \in O(\{\lambda_0\}, Z) \) with \( (\lambda - B)f_i(\lambda) = 0 \) near \( \lambda_0 \) and \( z \in LH\{f_1(\lambda_0), f_2(\lambda_0), \ldots, f_n(\lambda_0)\} \).

Although the case \( n = 1 \) is well known to the specialists, we indicate a possible proof. Since \( \lambda_0 \in \rho_c(B) \), there is a finite-dimensional subspace \( M \) of \( Z \) such that \( Z = (\lambda_0 - B)Z \oplus M \). Let \( \{z_1, \ldots, z_r\} \) be a basis for \( \text{Ker}(\lambda_0 - B) \). The map \( (\lambda_0 - B, i) : Z \oplus M \to Z \), where \( i \) denotes the inclusion map of \( M \) into \( Z \), is onto for \( \lambda = \lambda_0 \). By [10, Lemma 1.7] one can choose analytic functions \( g_1, \ldots, g_r \in O(\{\lambda_0\}, Z \oplus M) \) with \( (\lambda - B, i)g_i(\lambda) = 0 \) near \( \lambda_0 \) and \( g_i(\lambda_0) = z_i \). By [17, Lemma 2.1] the sequence

\[
0 \to \mathbb{C}^r \xrightarrow{\psi(\lambda)} Z \oplus M \xrightarrow{(\lambda - B, i)} Z \to 0,
\]

is exact for \( \lambda \) near \( \lambda_0 \). If \( \lambda_0 \notin \Sigma_H \), then \( \text{Im}(\lambda - B) \cap M = \{0\} \) for \( \lambda \) near \( \lambda_0 \), hence the functions \( g_i \) \((i = 1, \ldots, r)\) have values in \( Z \) near \( \lambda_0 \).

Next we assume that the assertion is true for \( n - 1 \) and consider an element \( z \in \text{Ker}(\lambda_0 - B)^n \setminus \text{Ker}(\lambda_0 - B)^{n-1} \). We choose an analytic function \( f \in O(\{\lambda_0\}, Z) \) with \( (\lambda - B)f(\lambda) = 0 \) near \( \lambda_0 \) and \( f(\lambda_0) = (\lambda_0 - B)^{n-1}z \). Then

\[
(B - \lambda)f^{(j)}(\lambda) = jf^{(j-1)}(\lambda)
\]

for \( \lambda \) near \( \lambda_0 \) and \( j \geq 1 \) and, in particular,

\[
(B - \lambda_0)^{n-1}\left(z - \frac{1}{(n-1)!}f^{(n-1)}(\lambda_0)\right) = 0.
\]

The induction is completed by applying the induction hypothesis.

Let \( f \in O(\{\lambda_0\}, Z) \) be a function with \( (\lambda - B)f(\lambda) = 0 \) and let \( k \in \{0, \ldots, n - 1\} \). The observation that

\[
\sum_{j=0}^{n-1}\frac{h^j(\lambda)}{j!}(B - \lambda)^j f^{(k)}(\lambda) = \sum_{j=0}^{k}\binom{k}{j}h^{(j)}(\lambda)f^{(k-j)}(\lambda) = C'f^{(k)}(\lambda)
\]

holds for all \( \lambda \), concludes the proof.

Let \( T \in L(X) \) be a continuous linear operator on a complex Banach space \( X \). As usual we denote by \( \sigma_\delta(T) \) the defect spectrum of \( T \), i.e., the set of all points \( \lambda \in \mathbb{C} \) for which \( \lambda - T \) is not onto. For a closed set \( F \) in \( \mathbb{C} \) the spectral subspace of \( T \) belonging to \( F \) is by definition the linear space

\[
X_T(F) = \{x \in X; \ x \in (z - T)O(\mathbb{C} \setminus F, X)\}.
\]

Proposition 1.3. For \( M \subset \rho_c(A) \) arbitrary the set

\[
Y(M) = \bigcap (\lambda - A)^nY; \ \lambda \in M, \ n \in \mathbb{N}
\]

is a \( \sigma(Y, Z) \)-closed space in \( \text{Lat}(A) \) with \( \sigma_\delta(A|Y(M)) \subset \mathbb{C} \setminus M \).
Proof. Since products of Fredholm operators are Fredholm, the spaces \((\lambda - A)^{n_1}Y \oplus \cdots \oplus (\lambda - A)^{n_k}Y\) occurring above are norm-closed, hence also \(\sigma(Y, Z)\)-closed [12, §33.4.1], invariant subspaces for \(A\).

Since for any choice of pairwise distinct complex numbers \(\lambda_1, \ldots, \lambda_k\) and nonnegative integers \(n_1, \ldots, n_k\) the identity
\[
(\lambda_1 - A)^{n_1}Y \cap \cdots \cap (\lambda_k - A)^{n_k}Y = (\lambda_1 - A)^{n_1} \cdots (\lambda_k - A)^{n_k}Y
\]
holds [11, Aufgabe 80.5], we obtain the description
\[
Y(M) = \bigcap ((\lambda_1 - A) \cdots (\lambda_n - A)Y ; \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in M).
\]

We fix a point \(\lambda_0 \in M\) and define \(N = \ker(\lambda_0 - A)\). We denote by \(\mathcal{F}\) the set of all functions \(f: M \to \mathbb{N}\), which are equal to zero almost everywhere and set \(Y_f = \bigcap_{\lambda \in M} (\lambda - A)^{(2)}Y\) for \(f \in \mathcal{F}\). Since \(N\) is finite-dimensional, there is an element \(f_0 \in \mathcal{F}\) with \(Y_{f_0} \cap N = Y(M) \cap N\). If \(y \in Y(M)\), then for each \(f \in \mathcal{F}\) there is an element \(x_f \in Y_f\) with \((\lambda_0 - A)x_f = y\). Since for \(f \in \mathcal{F}\) with \(f \geq f_0\)
\[
x_{f_0} = x_f + (x_{f_0} - x_f) \in Y_f + (Y_{f_0} \cap N) \subset Y_f,
\]
we conclude that \(x_{f_0} \in Y(M)\).

Using Proposition 1.3 it is easy to show that the spectral subspaces \(X_T(F)\) of an operator \(T \in L(X)\) belonging to a closed set \(F\) with \(\sigma_e(T) \subset F\) are closed.

**Corollary 1.4.** For each closed set \(F\) in \(\mathbb{C}\) with \(\sigma_e(A) \subset F\) the identity \(Y_A(F) = Y(C \setminus F)\) holds.

**Proof.** It is well known that for an arbitrary closed set \(F\) in \(\mathbb{C}\) the left space is contained in the right. The reason is, that for each function \(f \in O(C \setminus F, Y)\) for which \((\lambda - A)f(\lambda)\) is constant on \(C \setminus F\) and each \(\mu \in C \setminus F\), the unique function \(g \in O(C \setminus F, Y)\) with \(g(\lambda) = (f(\lambda) - f(\mu))/(\mu - \lambda), \ \lambda \neq \mu\), satisfies \((\lambda - A)g(\lambda) = f(\mu)\) on \(C \setminus F\).

In Proposition 1.3 we have shown that \(\lambda - A: Y(C \setminus F) \to Y(C \setminus F)\) is onto for each \(\lambda \in C \setminus F\). Therefore (see, e.g., [13, Theorem 5.1]) for each \(y \in Y(C \setminus F)\) there is an analytic function \(f \in \sigma(C \setminus F, Y)\) with \(y = (\lambda - A)f(\lambda)\) on \(C \setminus F\).

If \(T \in L(X)\) is a continuous linear operator on a complex Banach space \(X\), then for each open set \(U\) in \(\mathbb{C}\) we define \(X_T(U) = \cup_K X_T(K)\), where \(K\) ranges over all compact subsets of \(U\). As a consequence of Corollary 1.4 one obtains a duality relation between the spectral subspaces of \(A\) and \(B\).

**Corollary 1.5.** For each closed set \(F\) with \(\sigma_e(A) \subset F\) we have
\[
Y_A(F) = \perp Z_B(C \setminus F).
\]

**Proof.** Again it is well known that for an arbitrary closed set \(F\) in \(\mathbb{C}\) the left space is contained in the right. The idea is the following. For \(y \in Y_A(F)\) and \(z \in Z_B(C \setminus F)\) one can choose a compact subset \(K\) of \(C \setminus F\) and functions \(f \in O(C \setminus F, Y), \ g \in O(C \setminus K, Z)\) with \(y = (\lambda - A)f(\lambda)\) on \(C \setminus F\), \(z = (\lambda - B)g(\lambda)\) on \(C \setminus K\). If \(\Gamma\) is a cycle that surrounds \(K\) in \(C \setminus K\), then Cauchy's integral theorem implies that
\[
\langle y, z \rangle = \frac{1}{2\pi i} \int_\Gamma \langle y, g(\lambda) \rangle d\lambda = \frac{1}{2\pi i} \int_\Gamma \langle f(\lambda), z \rangle d\lambda = 0.
\]
If $F \supset \sigma_e(A)$, then using Corollary 1.4 one obtains

$$\perp Z_B(C\setminus F) \subset \bigg( \bigvee_{\lambda \in C\setminus F} \ker(\lambda - B)^n \bigg) \bigcap \bigg( \bigcup_{\lambda \in C\setminus F} \im(\lambda - A)^n \bigg) = Y_A(F).$$

Here the first inclusion follows from the fact that the generalized eigenspaces $\ker(\lambda - B)^n$, $\lambda \in C\setminus F$, $n \geq 1$, are contained in $Z_B(\{\lambda\})$.

The last two results together with Proposition 1.2 allow a precise description of the kernel of the representation $\Phi_H$. As before let $H \subset \sigma(A)$ be a hole of $\sigma_c(A)$. We define $H_0 = H\setminus \Sigma_H$ and denote by $(A)'$ the commutant of $A$ in $L(Y)$.

**Theorem 1.6.** (a) If $\ind_H(A) \leq 0$, then $\Phi_H$ is a continuous algebra homomorphism with

$$\ker \Phi_H = \{ C \in \sigma\text{-AlgLat}(A) ; CY \subset Y_A(C\setminus H_0) \}.$$

For $C \in \sigma\text{-AlgLat}(A)$ and each $\sigma(Y, Z)$-continuous operator $U \in (A)'$ we have $\im(CU - UC) \subset Y_A(C\setminus H_0)$.

(b) If $\ind_H(A) > 0$, then $\Phi_H$ is a continuous algebra homomorphism with

$$\ker \Phi_H = \{ C \in \sigma\text{-AlgLat}(A) ; Y_A(H_0) \subset \ker C \}.$$

For $C \in \sigma\text{-AlgLat}(A)$ and each $\sigma(Y, Z)$-continuous operator $U \in (A)'$ we have $Y_A(H_0) \subset \ker(CU - UC)$.

**Proof.** (a) Assume that $\ind_H(A) \leq 0$. We fix an element $C \in \sigma\text{-AlgLat}(A)$ and set $h = \Phi_H(C)$. By Proposition 1.2 we know that $h = 0$ if and only if $\ker(\lambda - B)^n \subset \ker C'$ for all $\lambda \in H_0$ and $n \geq 1$ or, equivalently, $\im C \subset \im(\lambda - A)^n$ for all $\lambda \in H_0$ and $n \geq 1$. Therefore the claimed representation of $\ker \Phi_H$ follows from Corollary 1.4. If $C \in \sigma\text{-AlgLat}(A)$ and $U \in (A)'$ is $\sigma(Y, Z)$-continuous, then again by Proposition 1.2 it follows that $\ker(\lambda - B)^n \subset \ker(CU - UC)'$ for all $\lambda \in H_0$ and $n \geq 1$.

(b) If $\ind_H(A) > 0$, then it suffices to apply part (a) to $B$ and to use Corollary 1.5.

**2. Main results**

In the sequel we shall make the simplifying assumption that the index of $A$ has the same sign on all holes of $\sigma_c(A)$. More precisely, we shall assume that $\ind_H(A) \leq 0$ for all holes $H$ in $\sigma_c(A)$. We shall denote by $U$ the set

$$U = \bigcup(H\setminus \Sigma_H ; H \subset \sigma(A) \text{ is a hole of } \sigma_c(A)).$$

Our assumption is satisfied, for instance, if the operator $A$ satisfies the single valued extension property, i.e., the map

$$O(W, Y) \rightarrow O(W, Y), \quad f \rightarrow (z - A)f$$

is one-to-one for all open subsets $W$ of $\mathbb{C}$; namely, in this case a result of Finch [9, Theorem 9] implies that $\ind_H(A) \leq -1$ for each hole $H$ in $\sigma_c(A)$ contained in $\sigma(A)$. Moreover, since by general Fredholm theory either $H \subset \sigma_p(A)$ or
\( \sigma_p(A) \cap H \) is discrete in \( H \), the same result of Finch shows that in this case \( \Sigma_H = \sigma_p(A) \cap H \).

Let \( S \) be a closed subset of \( \mathbb{C} \). Recall that the operator \( A \) is said to possess Bishop's property \( (\beta) \) (modulo \( S \)), if the map

\[
O(W, Y) \to O(W, Y), \quad f \to (z - A)f
\]

is injective with closed range for each open subset \( W \) of \( \mathbb{C} \) (resp. \( \mathbb{C} \setminus S \)).

Finally, as usual, if \( K \) is a compact set in \( \mathbb{C} \) and \( V \) is a bounded open set in \( \mathbb{C} \), then we shall say that \( K \) is dominating in \( V \) if

\[
\|f\|_{\infty, V} = \sup_{z \in K \cap V} |f(z)|
\]

holds for all bounded analytic functions \( f \) on \( V \). For operators satisfying Bishop's property \( (\beta) \) it was shown in [8] how to construct a representation of \( \text{AlgLat}(T) \) on the largest open set \( V \) in \( \mathbb{C} \) in which the essential spectrum of \( T \) is dominating. Our next aim is to show that the "Fredholm representations" constructed in Lemma 1.1 and the "Scott Brown representations" constructed in [8] are compatible with each other.

Let \( S \) be a closed set in \( \mathbb{C} \) such that our given operator \( A \in \text{L}(Y) \) satisfies Bishop's property \( (\beta) \) modulo \( S \), and let \( V \subset \mathbb{C} \) be open such that \( \sigma_e(A) \) is dominating in \( V \) and \( S \cap V = \emptyset \) or \( S \subset V \). In [1] (for the construction and notation, see [8, §1]) it was shown how to construct a canonical \( S \)-decomposable lifting \((Z, B) \xrightarrow{q} (X, T)\) for \( B \). We recall that an operator \( T \in \text{L}(X) \) on a Banach space \( X \) is called \( S \)-decomposable if for each open cover \( C = U_0 \cup \cdots \cup U_n \) with \( S \subset U_0 \) there are spaces \( X_0, \ldots, X_n \in \text{Lat}(T) \) with \( X = X_0 + \cdots + X_n \), \( \sigma(T|X_i) \subset U_i \) \( (i = 0, \ldots, n) \).

The canonical lifting was used in §3 of [8] to construct a continuous algebra homomorphism

\[
\Phi_V : \sigma\text{-AlgLat}(A) \to H^\infty(V).
\]

**Lemma 2.1.** If \( C \in \sigma\text{-AlgLat}(A) \) and \( H \subset \sigma(A) \) is a hole in \( \sigma_e(A) \), then

\[
\Phi_H(C)|_{H \cap V} = \Phi_V(C)|_{H \cap V}.
\]

**Proof.** We fix a point \( \lambda \in H \cap V \) and define \( h = \Phi_H(C) \), \( g = \Phi_V(C) \). By our assumption that \( \text{ind}_H(A) \leq 0 \) we can choose a nonzero vector \( z \in \text{Ker}(\lambda - B) \). The construction of the canonical lifting \( T \) (see [8, §1]) guarantees that there is a vector \( x \in \text{Ker}(\lambda - T) \) with \( qx = z \). By Lemma 3.3 of [8] it follows that for all \( y \in Y \)

\[
\langle y, h(\lambda)z \rangle = \langle Cy, qx \rangle = y \otimes x(g) = \langle y, qg(T|X_T(\{\lambda\}))x \rangle = \langle y, g(\lambda)z \rangle.
\]

In view of the last lemma it is obvious that the representations \( \Phi_H \) and \( \Phi_V \), where \( H \) runs through all holes in \( \sigma_e(A) \) with \( H \subset \sigma(A) \), can be glued together to give a continuous algebra homomorphism

\[
\Phi : \sigma\text{-AlgLat}(A) \to H^\infty(\Omega),
\]

where \( \Omega = U \cup V \).
Theorem 2.2. The map $\Phi: \sigma\text{-AlgLat}(A) \to H^\infty(\Omega)$ is a continuous algebra homomorphism with

$$\text{Ker} \Phi = \{C \in \sigma\text{-AlgLat}(A); \text{Im} C \subset Y_A(C\setminus\Omega)\}.$$ 

For $C \in \sigma\text{-AlgLat}(A)$ and each $\sigma(Y, Z)$-continuous operator $U \in (A)'$ we have

$$\text{Im}(CU - UC) \subset Y_A(C\setminus\Omega).$$

Proof. Whenever $H \subset \sigma(A)$ is a hole of $\sigma(A)$ with $H_0 \cap (C\setminus V) \neq \emptyset$, each nonempty component of $H_0 \cap V$ has a nontrivial intersection with $C \setminus S$. This observation easily gives rise to the identity

$$Y_A(C\setminus\Omega) = Y_A(C\setminus U) \cap Y_A(C\setminus V).$$

We fix an element $C \in \sigma\text{-AlgLat}(A)$ as well as a $\sigma(Y, Z)$-continuous operator $U \in (A)'$ and define $g = \Phi(C)$. By Proposition 1.2 we know that $g|_U = 0$ if and only if $\text{Ker}(\lambda - B)^n \subset \text{Ker} C'$ for each $\lambda \in U$ and each $n \geq 1$ or, equivalently, if $\text{Im} C \subset \text{Im}(\lambda - A)^n$ for all $\lambda \in U$ and $n \geq 1$. Using Corollary 1.4 and Theorem 3.4 of [8] we obtain the claimed characterization of $\text{Ker} \Phi$.

Similarly, by Proposition 1.2 it follows that

$$\text{Ker}(\lambda - B)^n \subset \text{Ker}(C'U' - U'C') \quad (\lambda \in U, \ n \geq 1)$$

and hence that

$$\text{Im}(CU - UC) \subset \bigcap_{\lambda \in U, \ n \geq 1} \text{Im}(\lambda - A)^n = Y_A(C\setminus U).$$

The inclusion $\text{Im}(CU - UC) \subset Y_A(C\setminus V)$ follows from the proof of Lemma 3.7 of [8] (see also [8, proof of Theorem 3.4]).

As an application we obtain the results announced in the introduction. If $A$ satisfies Bishop’s property ($\beta$) globally, then $V$ can be chosen as the largest bounded open set in $C$, in which $\sigma_e(A)$ is dominating. The resulting set $\Omega$ is rather large in this case. More precisely,

$$\sigma(A) \cap (C\setminus\Omega) \subset (\sigma_e(A) \cap (C\setminus\Omega)) \cup (\sigma_p(A) \cap \rho_e(A)).$$

The first set $K = \sigma_e(A) \cap (C\setminus\Omega)$ is a subset of $\partial \sigma_e(A)$, which is dominating in no open subset of $C$. In particular, $R(K) = C(K)$ (cf. [3, Theorem 3]). The second set $N = \sigma_p(A) \cap \rho_e(A)$ is countable with all limit points contained in $\partial \sigma_e(A)$.

We recall from [1] that property ($\beta$) admits a dual characterization. A Banach space operator $T \in L(X)$ is said to possess property ($\delta$), if the map

$$O(W)' \hat{\circ} X \to O(W)' \hat{\circ} X, \quad u \to (z - T)u,$$

is onto for each open set $W$ in $C$ or, equivalently, if

$$X = X_T(\bar{U}_1) + \cdots + X_T(\bar{U}_n)$$

holds for each open cover $C = U_1 \cup \cdots \cup U_n$ (see [1] for the equivalence and other characterizations).
Corollary 2.3. Let $R \in L(E)$ be a continuous operator on a complex Banach space $E$.

(a) If $R$ satisfies property $(\beta)$ and
\[ \sigma_c(R) \cap \rho_c(R) = \emptyset, \quad E_R(\partial \sigma_c(R)) = \{0\}, \]
then $\text{AlgLat}(R) \subset (R)'$.

(b) If $R$ satisfies property $(\delta)$ and
\[ \sigma_c(R) \cap \rho_c(R) = \emptyset, \quad E_R(\mathbb{C} \setminus \partial \sigma_c(R)) = E, \]
then $\text{AlgLat}(R) \subset (R)'$.

Proof. As in [8, §1] we define
\[ Y = E, \quad Z = E', \quad A = R, \quad B = R' \]
in the setting of part (a) and
\[ Y = E', \quad Z = E, \quad A = R', \quad B = R \]
in the setting of part (b). Since property $(\beta)$ and property $(\delta)$ are completely dual to each other [1, §3], in both cases $A$ satisfies $(\beta)$ and $B$ satisfies $(\delta)$. Since for each closed set $F$ in $\mathbb{C}$
\[ Y_A(F) = Z_B(\mathbb{C} \setminus F) \quad [8, \text{Lemma 1.3}], \]
we have in both cases the relation $Y_A(\sigma(A) \cap (\mathbb{C} \setminus \Omega)) = \{0\}$. Thus, the assertions follow from Theorem 2.2.

Of course, Corollary 2.3 becomes wrong without the conditions
\[ \sigma_c(R) \cap \rho_c(R) = \emptyset \quad \text{[resp. } \sigma_c(R) \cap \rho_c(R) = \emptyset \text{].} \]
To see this, it suffices to recall that on a finite-dimensional space each operator $R$ with $\text{AlgLat}(R) \subset (R)'$ is reflexive [2].

Specialized to the case of hyponormal operators on Hilbert spaces we obtain the following consequences. We denote by $\lambda$ the planar Lebesgue measure.

Corollary 2.4. Let $A$ be a hyponormal operator on a Hilbert space $H$.

(a) If $\lambda(\partial \sigma_c(A)) = 0$, then $\text{AlgLat}(A)$ is commutative.

(b) If $A$ is pure and $\lambda(\partial \sigma_c(A)) = 0$, then $\text{AlgLat}(A) \subset (A)'$.

Proof. Recall that hyponormal operators satisfy Bishop's property $(\beta)$ [14, Theorem III.5.5]. If $\Omega$ is defined as above, then
\[ \sigma(A|H_A(\mathbb{C} \setminus \Omega)) \subset \sigma(A) \cap (\mathbb{C} \setminus \Omega) \subset \partial \sigma_c(A) \cup N. \]
Since hyponormal operators, the spectrum of which is of Lebesgue measure zero, are normal, the space $M = H_A(\mathbb{C} \setminus \Omega)$ is a reducing subspace for $A$ such that $A|_M$ is normal if $\lambda(\partial \sigma_c(A)) = 0$. Therefore, part (b) follows directly from Theorem 2.2. If $C, D \in \text{AlgLat}(A)$, then
\[ (CD - DC)(M^+) \subset M \cap M^+ = \{0\}. \]
Moreover, $(CD - DC)(M) = \{0\}$, since $A|_M$ is reflexive as a normal operator [6, Theorem II.8.5].

Since the above methods are of a comparatively general nature, it is perhaps not surprising that in cases where there is much more structure at hand they do not lead to the best possible results. By a result of Olin and Thomson [15] all subnormal operators are reflexive. As an application of our methods we only obtain:
Corollary 2.5. If $A$ is a subnormal operator on a Hilbert space, then $\text{AlgLat}(A)$ is commutative. If, in addition, $A$ is pure, then $\text{AlgLat}(A) \subset (A)^{''}$.

Proof. Since there is a reducing space $H_0$ for $A$ such that $A|_{H_0}$ is normal and $A|_{H_0}^+$ is pure and subnormal [6, Proposition III.2.1], it suffices to prove the second statement. But, if $A$ is pure, then $\sigma_p(A) = \emptyset$ and hence $\sigma(A|_{H_A}(\mathbb{C}\setminus\Omega)) \subset \sigma_e(A) \cap (\mathbb{C}\setminus\Omega)$, where the last set $K = \sigma_e(A) \cap (\mathbb{C}\setminus\Omega)$ satisfies $R(K) = C(K)$. But then $A|_{H_A(\mathbb{C}\setminus\Omega)}$ is normal [6, Theorem VI.1.1] and hence $H_A(\mathbb{C}\setminus\Omega) = \{0\}$.

References

8. J. Eschmeier and B. Prunaru, Invariant subspaces for operators with Bishop’s property ($\beta$) and thick spectrum, J. Funct. Anal. 94 (1990), 196–222.