MINIMAL RELATIVE RELATION MODULES OF FINITE $p$-GROUPS

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Abstract. Consider $1 \rightarrow S \rightarrow E \rightarrow G \rightarrow 1$, where $G$ is a finite $p$-group generated by $g_i$, $1 \leq i \leq d$, and $E$ a free product of cyclic groups $(g_i)$, $1 \leq i \leq d$. If $d$ is the minimum number of generators for $G$, then we prove that the largest elementary abelian $p$-quotient $S'/S''_p$, regarded as an $F_pG$-module via conjugation in $E$, is nonprojective and indecomposable.

The author [5] has introduced and studied relative relation modules. Consider

$$1 \rightarrow S \rightarrow E \overset{\psi}{\rightarrow} G \rightarrow 1,$$

where $G$ is a finite group generated by $g_i$, $1 \leq i \leq d$, $E$ the free product of any cyclic groups $(g_i)$, $1 \leq i \leq d$, and $e_i\psi = g_i$. Let $p$ be a (fixed) prime. The largest abelian $p$-quotient $\hat{S} = S/S''_p$, regarded as an $F_pG$-module via conjugation in $E$, is called the relative relation module (modulo $p$) of $G$ determined by $\psi$. If each $(e_i)$ is infinite, $\hat{S}$ is called a relation module of $G$. Gaschütz [1], Gruenberg [2, 3], and others have studied relation modules. $\hat{S}$ is called minimal if $G$ cannot be generated by fewer than $d$ elements. As a direct consequence of [3, Theorem (2.9)], minimal relation modules of $p$-groups are nonprojective and indecomposable. The aim of this paper is to prove

Theorem 1. If $|\langle e_i \rangle| = m_i |\langle g_i \rangle|$, $1 \leq m_i < \infty$, and $p \neq m_i$, $1 \leq i \leq d$, then the minimal relative relation module $\hat{S}$ of a $p$-group is nonprojective and indecomposable.

For the rest of the paper, let $G$ be a (finite) $p$-group and regard all modules as (right) $F_pG$-modules. It is a well-known fact that the Frattini subgroup of $G$ coincides with $G'G^p$, and hence the minimal number of generators of $G$ and $G/G'G^p$ is the same. Moreover, $F_pG$ and all its submodules are indecomposable, and $F_pG$ has only one irreducible module, namely, $F_p$. A minimal generating set for a module is an $F_pG$-generating set whose cardinality is less than or equal to any other generating set for the module. For a module $M$, define $[M,G]$ to be the span of $\{m(g-1)/m \in M, \ g \in G\}$, so that $M/[M,G]$ is the largest trivial quotient of $M$. We set $[M,G,G] = [[M,G],G]$. The following (well-known) result is not difficult to prove.

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Lemma 2. Let $H$ be any subgroup of $G$ and $M$ a module that affords the natural permutation representation of $G$ on the set of (right) cosets of $H$. Then $$[M, G]/[M, G, G] \cong G/HG'G''.$$ 

Corollary 3. Let $d$ be the minimum number of generators for $G$ and $M$ a module generated by $r$ elements. Then

- (a) $\dim(M/[M, G]) \leq r$,
- (b) $\dim([M, G]/[M, G, G]) \leq dr$, and
- (c) $\dim([M, G]/[M, G, G]) \leq d \dim M/[M, G]$.

Proof. (a) follows from the fact that the result is true for free modules of rank $r$, (b) follows by substituting $H = 1$ in Lemma 2, and (c) follows from (b) by observing that the minimal number of generators for $M$ is the same as the dimension of $M/[M, G]$.

Proof of Theorem 1. From [5, (2.13)] we obtain the following $\mathbb{F}_pG$-exact sequence:

1. $0 \to \mathcal{H} \to L \to M \to 0$

and

2. $0 \to M \to \bigoplus_{i=1}^{d} U_i \overset{\beta}{\to} \mathbb{F}_p \to 0,$

where $L$ is a free module of rank $d - 1$. Since $\mathcal{H}$ is a homomorphic image of the corresponding minimal relation module that is indecomposable and non-projective, $\mathcal{H}$ has no nonzero projective direct summand. It follows that (1) is a projective cover of $M$. By a theorem of Heller [4] the indecomposability of $\mathcal{H}$ will follow if we prove

Theorem. $M$ is indecomposable.

Proof. To prove this we use the following exact sequence (cf. [5, (2.13)]): $$0 \to M \to \bigoplus_{i=1}^{d} U_i \overset{\beta}{\to} \mathbb{F}_p \to 0,$$

where $U_i$ is the module that affords the natural permutation representation of $G$ on the cosets $\langle g_i \rangle$ and $u_i \beta = 1$, $1 \leq i \leq d$, where $u_i$ is an $\mathbb{F}_qG$-generator of $U_i$. By definition of $\beta$, the kernel $M$ of $\beta$ is generated by all $u_i - u_d$, $1 \leq i \leq d - 1$, and hence $\dim M/[M, G] \leq d - 1$. But $(M + [U, G])/[U, G]$ has dimension $d - 1$ and is a surjective image of $M/[M, G]$. Hence $[M, G] = [U, G] \cap M$, whence $[M, G] = [U, G] = \bigoplus_{i=1}^{d} [U_i, G]$, and also $\dim M/[M, G] = d - 1$. Now suppose that $M = M' \oplus M''$, and let $r = \dim(M'/[M', G])$. Since $[M, G] = [M', G] \oplus [M'', G] = \bigoplus_{i=1}^{d} [U_i, G]$ with $[U_i, G]$ indecomposable, by the Krull-Schmid theorem, $[M', G]$ is isomorphic to the direct sum of $s$, say, copies of $[U_i, G]$, and $[M'', G]$ is isomorphic to the direct sum of $r-s$ copies of $[U_i, G]$. By Lemma 2, $\dim([U_i, G]/[U_i, G, G]) = d - 1$ and so $$\dim([M', G]/[M', G, G]) = s(d - 1)$$
and
\[ \dim([M'', G]/[M'', G, G]) = (d - s)(d - 1). \]

By Corollary 3(b), however, \( s(d - 1) \leq dr \) and \( (d - s)(d - 1) \leq d(d - 1 - r) \).

Since these two inequalities sum to an equality, both of them must be equalities. But then \( d - 1 \) divides \( r \), which is only possible when either \( r = 0 \) or \( r = d - 1 \).

Thus either \( M' = 0 \) or \( M'' = 0 \), which completes the proof.

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REFERENCES


