ON CLOSED SUBSPACES OF OPERATOR RANGES

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Abstract. Necessary and sufficient for the closure of a linear subspace to lie in the range of a bounded linear operator is a certain “bounded preimage property” for the operator.

If \( T: X \to Y \) is a bounded linear operator between normed spaces then we shall, par abus de notation, also write [3]

\[
T: l_\infty(X) \to l_\infty(Y)
\]

for the operator induced between the corresponding spaces of bounded vector-valued sequences

\[
l_\infty(X) = \left\{ x \in X^N : \sup_n \| x_n \| < \infty \right\}.
\]

1. Theorem. If \( T \in \text{BL}(X, Y) \) is a bounded linear operator between Banach spaces and if \( M \subseteq Y \) is a linear subspace, then there is equivalence

\[
\text{cl} \: M \subseteq T(X) \iff l_\infty(M) \subseteq Tl_\infty(X).
\]

Proof. We shall show forward implication for complete \( X \) and backward implication for complete \( Y \). Whether or not either space is complete, the right-hand side of (1.1) is equivalent to

\[
T_{M}^{-1}: T_{-1}(M)/T_{-1}(0) \to Y \text{ bounded below}.
\]

Indeed if (1.2) holds then there is \( k > 0 \) for which

\[
dist(x, T_{-1}(0)) \leq k \| Tx \| \quad \text{for each } x \in T_{-1}(M),
\]

so that if \( y \in l_\infty(M) \) is arbitrary then there is \( x \in X^N \) for which

\[
y = Tx \quad \text{with dist}(x_n, T_{-1}(0)) \leq k \| y_n \|,
\]

and then \( z \in T_{-1}(0)^N \) for which

\[
\| x - z \| \leq 2 \text{dist}(x, T_{-1}(0)),
\]

giving

\[
y = T(x - z) \quad \text{with } x - z \in l_\infty(X).
\]
Conversely if (1.2) fails then there is $x \in X^N$ for which
\[ T x_n \in M, \quad \|T x_n\| \to 0, \quad \text{dist}(x_n, T^{-1}(0)) \geq 1. \]
Now with
\[ x'_n = \begin{cases} \|T x_n\|^{-1/2} x_n & \text{if } T x_n \neq 0, \\ n x_n & \text{if } T x_n = 0, \end{cases} \]
we have $\|T x'_n\| \to 0$ and $\text{dist}(x'_n, T^{-1}(0)) \to \infty$ so that
\[ T x' \in c_0(M) \subseteq l_\infty(M) \quad \text{and} \quad T x' \not\subseteq T l_\infty(X). \]

If, in particular, the spaces $X$ and $Y$ are complete then condition (1.2) is also equivalent to the left-hand side of (1.1). To see this we need an auxiliary subspace
\[ M^\sim = T \text{cl} T^{-1}(M). \]
Evidently
\[ M \subseteq M^\sim \subseteq T(X) \cap \text{cl} M, \]
and hence, in particular,
\[ T^\sim_M \text{ bounded below} \iff T^\sim_M \text{ bounded below.} \]
The operator $T^\sim_M$ is one-to-one, with range $M^\sim$, and if $X$ is complete defined on the complete space
\[ T^{-1}(M^\sim)/T^{-1}(0) = \text{cl} T^{-1}(M)/T^{-1}(0), \]
so that
\[ T^\sim_M \text{ bounded below} \Rightarrow M^\sim = \text{cl} M^\sim \]
since $M^\sim$ is complete. By (1.6) this gives
\[ M^\sim = \text{cl} M, \]
and hence also the left-hand side of (1.1) holds. Conversely if this happens then $\text{cl} M$ is complete (if $Y$ is) and the open mapping theorem gives
\[ T^\sim_{\text{cl} M}: T^{-1}(\text{cl} M)/T^{-1}(0) \to Y \text{ bounded below}, \]
and hence also (1.2). \(\square\)

The same argument gives the analogue of Theorem 1 in which the right-hand side of (1.1) is replaced by the corresponding property for subsets
\[ \beta(M) \subseteq T \beta(X), \]
where $\beta(X)$ denotes the bounded subsets of $X$; an easy consequence is that compact operators on complete spaces have the "Calkin property" [4; 2, Theorem III.1.12]
\[ \text{cl} M \subseteq T(X) \Rightarrow M \text{ finite dimensional.} \]

Notice that we have proved two versions of Theorem 1: we also have
\[ \text{cl} M \subseteq T(X) \Leftrightarrow c_0(M) \subseteq T l_\infty(X). \]
In the particular case $M = T(X)$ Albrecht and Mehta [1, Lemma 2.1] have shown that also
\[ \text{cl} M \subseteq T(X) \Leftrightarrow l_\infty(M) \subseteq T(X) + c_0(Y), \]
which says that the image of $M$ in the "enlargement" of $Y$ [3, Definition 1.9.2] is included in the range of the enlargement of $T$. 

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