ON IMMERSIONS OF $k$-CONNECTED $n$-MANIFOLDS

B. H. LI AND M. E. MAHOWALD

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Abstract. In this note we classify up to regular homotopy the classes of immersions of $k$-connected closed differentiable $n$-manifolds in $\mathbb{R}^{2n-k}$.

In [W] it was claimed that for a $k$-connected closed differentiable manifold $M$ of dimension $n$ with $0 < 2k < n-2$, the regular homotopy classes of $M$ in $\mathbb{R}^{2n-k}$ are in one to one correspondence with $\pi_n(\nu_{2n-k,n})$, where $\nu_{2n-k,n}$ is a Stiefel manifold. But this result is incorrect. In this note, we will present the correct answer and give counterexamples to the claim of [W].

To state our result, some definitions are needed. First, let $\pi_i(M)$ be the $i$th homotopy group of $M$. Then

$$h : \pi_{k+1}(M) \to \widetilde{KO}(S^{k+1})$$

is defined as follows: If $\alpha \in \pi_{k+1}(M)$ is represented by a map $\tilde{\alpha} : S^{k+1} \to M$, then $h(\alpha)$ is the element in $\widetilde{KO}(S^{k+1})$ represented by $2\tilde{\alpha}^*\nu_M$, where $\nu_M$ is the stable normal bundle of $M$. It is easy to see that $h$ is well defined and is a homomorphism.

Next, let

$$k : \widetilde{KO}(S^{k+1}) \to \pi_k(SO)$$

be the natural isomorphism,

$$J : \pi_k(SO) \to \pi_k^+ = \pi_n(S^{n-k})$$

be the $J$-homomorphism, and let

$$S^{n-k} \xrightarrow{i} V_{2n-k,n} \to V_{2n-k,n-1}$$

be the natural fibration. Let

$$e = i_* \circ J \circ k \circ h : \pi_{k+1}(M) \to \pi_n(\nu_{2n-k,n}).$$

Then we have

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Theorem. If \( 0 \leq 2k \leq n - 2 \) and \( M \) is a \( k \)-connected closed differentiable manifold, then the regular homotopy classes of immersions of \( M \) in \( \mathbb{R}^{2n-k} \) are in one-to-one correspondence with the cokernel of \( e \).

Example. Let \( K^2 \) be the Cayley projective plane and \( M = K^2 \times S^{23} \). Then \( M \) is 7-connected with dimension 39, and the homotopy sequence of the fibration \( S^{32} \to V_{71,39} \to V_{71,38} \) gives the split exact sequence

\[
0 \to \pi_{39}(S^{32}) \overset{i_*}{\to} \pi_{39}(V_{71,39}) \to \pi_{39}(V_{71,38}) \to 0
\]

\[
\begin{array}{c}
\pi_{39}(S^{32}) \\
\cong \mathbb{Z}_{240}
\end{array}
\quad
\begin{array}{c}
\pi_{39}(V_{71,39}) \\
\cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{16}
\end{array}
\]

Now, the tangent bundle of \( K^2 \) restricted to \( K^1 \cong S^8 \) is stably equivalent to the canonical 8-dimensional vector bundle, which can be defined by regarding \( S^8 \cong \mathbb{R}^8 \cup \{\infty\} = D_+ \cup D_- \) and \( D_\alpha \) as the Cayley algebra. Let \( D_+ = \{x \in \mathbb{R}^8/||x|| \leq 1\} \) and \( D_- = S^8-D_+ \). Glue \( D_+ \times \mathbb{R}^8 \) and \( D_- \times \mathbb{R}^8 \) over \( D_+ \cap D_- = S^7 \) by \((x,y) \sim (x,xy)\) where \( xy \) is the product of \( x \in S^7 \) and \( y \in \mathbb{R}^8 \) as Cayley numbers (see Steenrod [S, p. 109]).

Let \( 1, e_1, \ldots, e_7 \) be the units of Cayley algebra. Then \( x \to (x, x e_1, \ldots, x e_7) \) defines a map \( S^7 \to SO(8) \), which represents a generator of the first summand of \( \pi_7(SO(8)) \cong \pi_7(S^7) \oplus \pi_7(SO(7)) \cong \mathbb{Z} \oplus \mathbb{Z} \), hence a generator of \( \pi_7(SO) \) (cf. [L, Remark 4]). Thus \( 2T(M) \mid K \) represents twice a generator of \( KO(S^8) \). Since \( J = \pi_7(SO) \to \pi_5 \) is surjective, we see that the cokernel of \( e \) is isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{16} \). This shows that the result of [W] is incorrect.

We can produce more examples. Let \( \xi \) be any 8-dimensional vector bundle over \( S^8 \). Then the second Pontriagin class \( p_2(\xi) = 6q \) for some \( q \in \mathbb{Z} \) (cf. [K]). Let \( M = S(\xi \oplus 1) \times S^{23} \), where \( S(\xi \oplus 1) \) is the total space of the sphere bundle of \( (\xi \oplus 1) \). Then \( \pi_6(M) = \mathbb{Z} \oplus \mathbb{Z} \) and \( h \) maps one summand to 0, another to \( 2q\mathbb{Z} \). Thus the image of \( e \) can be any subgroup of \( \pi_5 \cong \mathbb{Z}_{240} \) consisting of even elements, and only when \( q \equiv 0 \mod 120 \) is the result of [W] true.

Since \( 2KO(S^{k+1}) = 0 \), if \( k \neq 3 \), \( 7 \mod 8 \), we have

Corollary 1. If \( k \neq 3 \), \( 7 \mod 8 \), then the cokernel of \( e \) is \( \pi(V_{2n-k, n}) \).

If \( n-k \) is odd, then \( i_* = \pi_n(S^{n-k}) \to \pi_n(V_{2n-k, n}) \) sends \( 2\pi_n(S^{n-k}) \) to zero since the first two essential cells of the Stiefel manifold form a \( Z/2 \) Moore space. So we have

Corollary 2. If \( n-k \) is odd and \( k \neq 3 \), \( 7 \mod 8 \), then the image of \( e \) is zero.

Proof of the theorem. By using normal bordism theory, we see from [Ko] or [D] that \( \text{Imm}[M, \mathbb{R}^{2n-k}] \) (or \([M \times \mathbb{R}^{2n-k}]\)) \cong \( \Omega_k(M \times P^\infty, \phi) \), where \( P^\infty \) is the infinite real projective space, \( \phi = (2n-k)\lambda - \lambda \otimes T(M) - T(M) \), and \( \lambda \) is the canonical line bundle over \( P^\infty \). There is an exact sequence

\[
\Omega_{k+1}(M \times P^\infty, * \times P^\infty, \phi) \overset{\partial}{\to} \Omega_k(P^\infty, \phi|_{P^\infty}) \\
\to \Omega_k(M \times P^\infty, \phi) \to \Omega_k(M \times P^\infty, * \times P^\infty, \phi)
\]

(cf. [D, p. 310]), where \(* \in M \) is a point. Since \((M \times P^\infty, P^\infty)\) is \( k \)-connected, it follows from Proposition 5.1 in [D] that \( \Omega_k(M \times P^\infty, * \times P^\infty, \phi) = 0 \),
and the natural homomorphism
\[ \mu : \Omega_{k+1}(M \times P^\infty, *, \times P^\infty, \phi) \to H_{k+1}(M \times P^\infty, *, \times P^\infty, Z(\phi)) \]
is an isomorphism.

Now look at the exact sequence
\[ \rightarrow H_{k+1}(P^\infty, Z(\phi|_{P^\infty})) \xrightarrow{j} H_{k+1}(M \times P^\infty, *, \times P^\infty, z(\phi)) \]
\[ \xrightarrow{\partial} H_k(P^\infty, z(\phi|_{P^\infty})) \xrightarrow{i} H_k(M \times P^\infty, z(\phi)). \]

Since \( W_1(\phi) = W_1((n-k)\lambda) \), we have \( Z(\phi) = Z \otimes Z((n-k)\lambda) \). Hence, by the Künneth formula for twisted coefficients,
\[ H_*(M \times P^\infty, Z(\phi)) = H_*(P^\infty, Z((n-k)\lambda)) \oplus H_*(M, Z) \otimes H_0(P^\infty, Z((n-k)\lambda)) \]
for \(* \leq k+1\). This shows that \( i_k \) is an isomorphism and \( j \) is an isomorphism from \( H_{k+1}(M, Z) \otimes H_0(P^\infty, Z((n-k)\lambda)) \) to \( H_{k+1}(M \times P^\infty, P^\infty, Z(\phi)) \).

Now \( \Omega_k(P^\infty, \phi|_{P^\infty}) = \Omega_k(P^\infty, (n-k)\lambda) = \pi_n(V_{2n-k,n}) \) (cf. [D, Proposition 7.3] or [Ko, Proposition 5.4]). We need only to calculate the image of \( \partial \) in the following diagrams:
\[ \Omega_{k+1}(M \times P^\infty, P^\infty, \phi) \xrightarrow{\partial} \Omega_k(P^\infty, \phi|_{P^\infty}) \]
\[ H_{k+1}(M, Z) \otimes H_0(P^\infty, z((n-k)\lambda)) \xrightarrow{\pi_n(V_{2n-k,n})} \]

By the Hurewicz isomorphism theorem,
\[ H_{k+1}(M) \cong H_{k+1}(M, Z) \quad \text{if } k > 0, \]
\[ \frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]} \cong H_1(M, Z) \quad \text{if } k = 0. \]

Let \( S^{k+1} = D_+ \cup D_- \), where \( D_+ \) and \( D_- \) are disks with \( D_+ \cap D_- = S^k \), and \( c : S^{k+1} \to P^\infty \) be a constant map. For any \( \alpha \in \pi_{k+1}(M,*) \), we can choose \( \tilde{\alpha} : S^{k+1} \to M \), representing \( \alpha \) with \( \tilde{\alpha}(D_-) = * \).

Let \( \tilde{\alpha} = (\tilde{\alpha}, c) : D_+ \to M \times P^\infty \). Then \( \tilde{\alpha} \) maps \( (D_+, \partial D_+) \) into \( (M \times P^\infty, *, \times P^\infty) \). Regard \( \phi \) as a stable bundle; then \( T(D_+) \oplus \tilde{\alpha}^*\phi \) obviously has a trivialization \( V \). Thus the triple \( ((D_+, \partial D_+), \tilde{\alpha}, V) \) defines an element in \( \Omega_{k+1}(M \times P^\infty, *, \times P^\infty, \phi) \), and \( (\partial D_+, \tilde{\alpha}|_{\partial D_+}, V|_{\partial D_+}) \) defines an element in \( \Omega_k(\ast, \text{trivial}) \equiv \pi_k^* \), which can be regarded as an element of \( \Omega_k(P^\infty, \phi|_{P^\infty}) \) via the inclusion \( * = c(S^{k+1}) \subset P^\infty \). From the isomorphisms
\[ \Omega_{k+1}(M \times P^\infty, *, \times P^\infty, \phi) \cong H_{k+1}(M \times P^\infty, *, \times P^\infty, Z(\phi)) \]
\[ \cong H_{k+1}(M, Z(\phi)) \otimes H_0(P^\infty, Z((n-k)\lambda)) \]
\[ \cong \pi_{k+1}(M) \otimes H_0(P^\infty, Z((n-k)\lambda)). \]
We see that the image of
\[ \partial : \Omega_{k+1}(M \times P^\infty, *, \times P^\infty, \phi) \to \Omega_k(P^\infty, \phi|_{P^\infty}) \]
is exactly the set \{ [((\partial D_+, \tilde{\alpha}|_{\partial D_+}, V|_{\partial D_+})]/\alpha \in \pi_{k+1}(M) \}. 

Regard $V|_{\partial D_+}$ as a trivialization of
\[(T(D_-) \oplus \tilde{\alpha}^* \phi)|_{\partial D_-};\]
then it defines an element of $\pi_k(SO)$, which is the value of $k: \tilde{KO}(S^{k+1}) \to \pi_k(SO)$ on the stable bundle $\tilde{\alpha}^*(2\nu) = h(\alpha)$. It is obvious that the element in $\Omega_k(\ast, \text{trivial})$ defined by $(\partial D_+ , \tilde{\alpha}|_{\partial D_+} , V|_{\partial D_+})$ is $(J \circ k \circ h)(\alpha)$.

Now, the only remaining thing is to see that the diagram
\[
\Omega_k(\ast, \text{trivial}) \longrightarrow \Omega_k(P^\infty, (n-k)\lambda)
\]

\[
\begin{array}{ccc}
\pi_n(S^{n-k}) & \overset{i_*}{\longrightarrow} & \pi_n(V_{2n-k,n})
\end{array}
\]

commutes. First

\[
\Omega_k(P^\infty, (n-k)\lambda) \cong \Omega_k(p^{n-1}, (n-k)\lambda) \cong \pi_n\left(\frac{P^{2n-k-1}}{p^{n-k-1}}\right)
\]

where $P^{2n-k-1}/p^{n-k-1}$ is the Thom space of $(n-k)\lambda$. Under the natural map

\[
p^{2n-k-1}/p^{n-k-1} \to V_{2n-k,n}
\]

the Thom space of $(n-k)\lambda$ restricted to a point maps onto $i(S^{n-k}) \subset V_{2n-k,n}$. So the above diagram commutes. So far we have proved that in the following diagram

\[
\Omega_{k+1}(M \times P^\infty, \ast \times P^\infty, \phi) \overset{\partial}{\longrightarrow} \Omega_k(P^\infty, \phi|_{P^\infty}) \to \Omega_k(M \times P^\infty, \phi) \to 0
\]

\[
\begin{array}{ccc}
\pi_{k+1}(M) & \overset{e= i_* \circ k \circ h}{\longrightarrow} & \pi_n(V_{2n-k,n})
\end{array}
\]

the images of $\partial$ and $e$ are isomorphic and hence so is the theorem.

References


Institute of Systems Science, Academia Sinica, Beijing 100080, People’s Republic of China

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

E-mail address: mark@math.nwu.edu