A TRANSITIVITY THEOREM FOR ALGEBRAS OF ELEMENTARY OPERATORS

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Abstract. Let \( A \) be a C*-algebra and \( \mathcal{E} \) the algebra of all elementary operators on \( A \), and let \( a = (a_1, \ldots, a_n) \), \( b = (b_1, \ldots, b_n) \) \( \in A^n \). It is proved that \( b \) is contained in the closure of the set \( \{(Ea_1, \ldots, Ea_n) : E \in \mathcal{E}\} \) if and only if each complex linear combination \( \sum_{j=1}^{n} \lambda_j b_j \) is contained in the closed two-sided ideal generated by \( \sum_{j=1}^{n} \lambda_j a_j \). In particular, a bounded linear operator on \( A \) preserves all closed two-sided ideals if and only if it is in the strong closure of \( \mathcal{E} \).

1. Introduction, motivation, and notation

An elementary operator on a ring \( A \) is a map \( E : A \rightarrow A \) of the form

\[
E x = \sum_{i=1}^{m} u_i x v_i \quad (x \in A),
\]

where \( \bar{u} = (u_1, \ldots, u_m) \) and \( \bar{v} = (v_1, \ldots, v_m) \) are fixed \( m \)-tuples of elements of \( A \). In the last decade there has been considerable interest in such operators, especially in the cases when \( A \) is the algebra \( B(H) \) of all bounded operators on a Hilbert space \( H \), the Calkin algebra or a general prime C*-algebra (see \([1, 7, 8]\) and their bibliographies). In this note we will be concerned with algebras of elementary operators on general C*-algebras.

Our study here is motivated by the following classical algebraic considerations. For any unital algebra \( A \) (over some field) the set \( \mathcal{E} \) of all elementary operators on \( A \) is again an algebra (with the usual operations of addition, multiplication by scalars, and composition of operators). The algebra \( A \) itself can be regarded as a (left) module over \( \mathcal{E} \), the submodules of which are precisely the two-sided ideals of \( A \), and the module endomorphisms of \( A \) are just the multiplications by elements of the centre \( C \) of \( A \). For any positive integer \( n \) the direct sum \( A^n \) of \( n \) copies of \( A \) is then, of course, also an \( \mathcal{E} \)-module. Let us consider the following question.

Given \( \bar{a}, \bar{b} \in A^n \), under what conditions does there exist an elementary operator \( E \in \mathcal{E} \) such that \( E\bar{a} = \bar{b} \) (that is, \( Ea_j = b_j \) for \( j = 1, \ldots, n \))?
One necessary condition is obvious: since \( Ea_j = b_j \) \((j = 1, \ldots, n)\) implies that \( E(\sum_{j=1}^n \lambda_j a_j) = \sum_{j=1}^n \lambda_j b_j \) for arbitrary central elements \( \lambda_j \in \mathcal{A} \), we see that \( \sum_{j=1}^n \lambda_j b_j \) must be in the two-sided ideal generated by \( \sum_{j=1}^n \lambda_j a_j \). To shorten the notation, for every \( \vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \) and every \( \vec{x} = (x_1, \ldots, x_n) \) the linear combination \( \sum_{j=1}^n \lambda_j x_j \) will be denoted by \( \vec{\lambda} \cdot \vec{x} \). Also, let us agree that the word 'ideal' means a two-sided ideal and, for each \( x \in \mathcal{A} \), denote by \( \langle x \rangle \) the ideal generated by \( x \). We may now ask the following

**Question 1.** Is the condition that \( \vec{\lambda} \cdot \vec{b} \in \langle \vec{\lambda} \cdot \vec{a} \rangle \) for each \( \vec{\lambda} \in \mathbb{C}^n \) also sufficient for the existence of an elementary operator \( E \) on \( \mathcal{A} \) such that \( Ea = \vec{b} \)?

In the special case, when \( \mathcal{A} \) has no nontrivial ideals, the answer to the last question is affirmative. Namely, in this case the condition reduces to the requirement that \( \vec{\lambda} \cdot \vec{a} = 0 \) implies \( \vec{\lambda} \cdot \vec{b} = 0 \) (for each \( \vec{\lambda} \in \mathbb{C}^n \)), and since \( \mathcal{A} \) is a simple \( \mathbb{C} \)-module, the existence of \( E \in \mathbb{C} \) satisfying \( Ea = \vec{b} \) follows from the Jacobson density theorem \([11, p. 220]\). In general, however, the answer to Question 1 is negative, as is shown by the following example.

**Example.** Let \( \mathcal{H} \) be an infinite-dimensional (complex) vector space, \( \mathcal{L} \) the algebra of all linear operators on \( \mathcal{H} \), \( \mathcal{F} \) the ideal of all finite rank operators, and \( \mathcal{C} \in \mathcal{L} \) a fixed operator such that \( \mathcal{C} \not\in \mathcal{F} \) and \( \mathcal{C}^2 \in \mathcal{F} \). Let \( \mathcal{A} \) be the subalgebra of \( \mathcal{L} \) generated by \( \mathcal{F} \), \( \mathcal{C} \), and the identity operator \( 1 \), and put \( \vec{a} = (1, \mathcal{C}) \), \( \vec{b} = (1, \mathcal{C}^2) \). Then clearly the centre of \( \mathcal{A} \) consists of scalars only, and it is easy to verify that the only proper ideals in \( \mathcal{A} \) are \( 0 \), \( \mathcal{F} \), and the ideal generated by \( \mathcal{F} \cup \{ \mathcal{C} \} \). It follows that, with \( \vec{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \), the ideal \( \langle \vec{\lambda} \cdot \vec{a} \rangle = \langle \lambda_1 + \lambda_2 \mathcal{C} \rangle \) is proper only if \( \lambda_1 = 0 \), and the condition \( \vec{\lambda} \cdot \vec{b} \in \langle \vec{\lambda} \cdot \vec{a} \rangle \) is satisfied for all \( \vec{\lambda} \in \mathbb{C}^2 \). On the other hand, we shall now see that there is no elementary operator \( E \) on \( \mathcal{A} \) such that \( E\vec{a} = \vec{b} \).

Assume to the contrary, that there exist \( u_i, v_i \in \mathcal{A} \) \((i = 1, \ldots, m)\) such that

\[
\sum_{i=1}^m u_i 1 v_i = 1 \quad \text{and} \quad \sum_{i=1}^m u_i \mathcal{C} v_i = \mathcal{C}^2.
\]

Since the quotient algebra \( \mathcal{A} / \mathcal{F} \) is obviously commutative, the last two identities imply that \( \mathcal{C}^2 = \mathcal{C} \pmod{\mathcal{F}} \), but this is in contradiction with the fact that \( \mathcal{C} \not\in \mathcal{F} \), \( \mathcal{C}^2 \in \mathcal{F} \).

We are now going to say a few words about the analogy of the above purely algebraic question in the context of Banach algebras. Let \( \mathcal{A} \) be a complex unital Banach algebra, and denote by \( [x] \) the closed ideal generated by an element \( x \in \mathcal{A} \). Given a positive integer \( n \) and \( \vec{a}, \vec{b} \in \mathbb{C}^n \) we may now ask

**Question 2.** When does there exist a sequence of elementary operators \( E_k \) \((k = 1, 2, \ldots)\) on \( \mathcal{A} \) such that \( E_k \vec{a} \) converge to \( \vec{b} \)?

Obviously a necessary condition for the existence of such a sequence of elementary operators is that \( \vec{\lambda} \cdot \vec{b} \in [\vec{\lambda} \cdot \vec{a}] \) for each \( \vec{\lambda} \in \mathbb{C}^n \), but a simple modification of the example shows that this condition is not sufficient. (Namely, in the Example we replace \( \mathcal{L} \) by the algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded linear operators on a separable Hilbert space \( \mathcal{H} \), and \( \mathcal{F} \) by the ideal \( \mathcal{F}(\mathcal{H}) \) of all
compact operators; we choose \( c \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{H}(\mathcal{H}) \) so that \( c^2 \in \mathcal{H}(\mathcal{H}) \), and let \( \mathcal{A} \) be the Banach algebra generated by \( \mathcal{H}(\mathcal{H}), c \), and the identity operator.)

Concerning general Banach algebras, we state here only one simple result.

**Proposition 1.1.** Let \( \mathcal{A} \) be a complex unital Banach algebra, and assume that \( \bar{a} = (a_1, ..., a_n) \in \mathcal{A}^n \) is such that \( [\bar{\lambda} \cdot \bar{a}] = \mathcal{A} \) for every \( \bar{\lambda} \neq 0, \bar{\lambda} \in \mathbb{C}^n \). Then for every \( \bar{b} = (b_1, ..., b_n) \in \mathcal{A}^n \) there exists an elementary operator \( E \) on \( \mathcal{A} \) such that \( E\bar{a} = \bar{b} \).

**Proof.** The proof is by an induction on \( n \). The case \( n = 1 \) is trivial, so let \( n > 1 \). We must prove that \( \mathcal{B} = \mathcal{A}^n \), and for this it suffices to prove that there exists \( F_n = F \in \mathcal{E} \) satisfying

\[
F a_j = 0 \quad \text{for} \quad j = 1, ..., n-1 \quad \text{and} \quad \langle F a_n \rangle = \mathcal{A}.
\]

For then, by the same argument, there exists for each \( i = 1, ..., n \) an \( F_i \in \mathcal{E} \) satisfying \( F_i a_j = 0 \) for \( j \neq i \) and \( \langle F_i a_i \rangle = \mathcal{A} \), and choosing \( E_i \in \mathcal{E} \) so that \( E_i F_i a_i = b_i \) (which is possible, since \( \langle F_i a_i \rangle = \mathcal{A} \) ), we see that the operator \( E \triangleq \sum_{i=1}^{n} E_i F_i \) then satisfies \( E\bar{a} = \bar{b} \). To prove the existence of \( F \) satisfying (1), suppose, to the contrary, that no such \( F \) exists, and denote by \( \mathcal{N} \) the left ideal in \( \mathcal{E} \) consisting of all \( F \in \mathcal{E} \) that satisfy \( F(a_1, ..., a_{n-1}) = 0 \). Note that \( \mathcal{N} a_n \) is an \( \mathcal{E} \)-submodule of \( \mathcal{A} \) and hence, an ideal in \( \mathcal{A} \). Since by assumption no \( F \in \mathcal{E} \) satisfies (1), \( 1 \notin \mathcal{N} a_n \); therefore, \( \mathcal{N} a_n \) is contained in some proper maximal ideal \( \mathcal{M} \) of \( \mathcal{A} \). By the induction hypothesis we have \( \mathcal{E}(a_1, ..., a_{n-1}) = \mathcal{A}^{n-1} \). Since by \( \mathcal{N} a_n \subseteq \mathcal{M} \), the map

\[
\varphi: \mathcal{A}^{n-1} \to \mathcal{A} / \mathcal{M}, \quad \varphi(F a_1, ..., F a_{n-1}) \triangleq F a_n + \mathcal{M} \quad (F \in \mathcal{E})
\]

is a well-defined homomorphism of \( \mathcal{E} \)-modules. Let \( \varphi_1, ..., \varphi_{n-1} \) be the components of \( \varphi \) (that is, \( \varphi_i: \mathcal{A} \to \mathcal{A} / \mathcal{M} \) are \( \mathcal{E} \)-module homomorphisms such that \( \varphi_j(x_1, ..., x_{n-1}) = \sum_{j=1}^{n-1} \varphi_j(x_j) \) for each \( (x_1, ..., x_{n-1}) \in \mathcal{A}^{n-1} \)). For each \( j \) the kernel of \( \varphi_j \) must contain \( \mathcal{M} \) (since for each \( m \in \mathcal{M} \) we have \( \varphi_j(m) = \varphi_j(m1) = m \varphi_j(1) = 0 \) in \( \mathcal{A} / \mathcal{M} \)). Hence \( \varphi_j \) induces an endomorphism \( \lambda_j \) of the \( \mathcal{E} \)-module \( \mathcal{A} / \mathcal{M} \). Such an endomorphism is necessarily a multiplication by a central element of \( \mathcal{A} / \mathcal{M} \), but the centre of each simple algebra is a field and the only field among the complex Banach algebras is the field \( \mathbb{C} \) of all complex numbers. It follows that \( \lambda_j \in \mathbb{C} \) for each \( j \). By the definition of \( \varphi \) we now have

\[
a_n + \mathcal{M} = \varphi(a_1, ..., a_{n-1}) = \sum_{j=1}^{n-1} \lambda_j a_j + \mathcal{M}.
\]

Hence \( a_n - \sum_{j=1}^{n-1} \lambda_j a_j \in \mathcal{M} \), but this contradicts the assumption that \( [\bar{\lambda} \cdot \bar{a}] = \mathcal{A} \) for each nonzero \( \bar{\lambda} \in \mathbb{C}^n \).

In the remaining part of this note we confine our attention to \( C^* \)-algebras, where the necessary condition that \( \bar{\lambda} \cdot \bar{b} \in [\bar{\lambda} \cdot \bar{a}] \) for each \( \bar{\lambda} \in \mathbb{C}^n \) turns out to be sufficient also for \( \bar{b} \) to be in the norm closure of the set \( \mathcal{E} \bar{a} \). At the same time the analogous question in the context of von Neumann algebras is answered.
At the end we shall also consider the so-called range inclusion problem for
elementary operators on factors.

2. The case of $C^*$-algebras

**Theorem 2.1.** Let $\mathcal{A}$ be a $C^*$-algebra, $\mathcal{E}$ the algebra of all elementary operators on $\mathcal{A}$, $n$ a positive integer, and $\tilde{a}, \tilde{b} \in \mathcal{A}^n$. Then $\tilde{b}$ belongs to the norm closure $\overline{\mathcal{E} \tilde{a}}$ of $\mathcal{E} \tilde{a}$ if and only if for each $\tilde{\lambda} \in \mathbb{C}^n$ the element $\tilde{\lambda} \cdot \tilde{b}$ is contained in the closed ideal $[\tilde{\lambda} \cdot \tilde{a}]$ generated by $\tilde{\lambda} \cdot \tilde{a}$.

If $\mathcal{A}$ is a von Neumann algebra, then $\tilde{b}$ belongs to the closure $\overline{\mathcal{E} \tilde{a}}$ of $\mathcal{E} \tilde{a}$ in the weak operator topology of $\mathcal{A}^n$ if and only if for each $\tilde{\lambda} \in \mathbb{C}^n$ (where $\mathcal{C}$ is the centre of $\mathcal{A}$) the relation $\tilde{\lambda} \cdot \tilde{a} = 0$ implies $\tilde{\lambda} \cdot \tilde{b} = 0$.

For a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ we denote by $M_{m,n}(\mathcal{B})$ the set of all $m \times n$ matrices with entries from $\mathcal{B}$, and we let $M_n(\mathcal{B}) = M_{n,n}(\mathcal{B})$. Further, we identify $\mathcal{B}^n$ with $M_{n,1}(\mathcal{B})$ so that elements of $\mathcal{B}^n$ are regarded as operators from $\mathcal{B}$ to $\mathcal{B}^n$. To prove Theorem 2.1 we first need a generalization of the well-known fact that every weakly closed ideal in a von Neumann algebra is generated by a central projection [4, p. 443].

**Lemma 2.2.** Let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then for each right $\mathcal{B}$-submodule $\mathcal{M}$ in $\mathcal{B}^n$ which is closed in the weak operator topology there exists an idempotent $p \in M_n(\mathcal{B})$ such that $\mathcal{M} = p\mathcal{B}^n$. Moreover, if $\mathcal{M}$ is an $\mathcal{B}$-bimodule, then $p \in M_n(\mathcal{C})$, where $\mathcal{C}$ is the centre of $\mathcal{B}$.

**Proof.** Since $\mathcal{M}$ is a right $\mathcal{B}$-module, we have

$$\mathcal{M} = \mathcal{M} \mathcal{B} = \mathcal{M} M_{1,n}(\mathcal{B}) M_{n,1}(\mathcal{B}).$$

Observe that $\mathcal{M} M_{1,n}(\mathcal{B})$ is a right ideal in the von Neumann algebra $M_n(\mathcal{B})$. Hence there exists a projection $p \in M_n(\mathcal{B})$ such that $\mathcal{M} M_{1,n}(\mathcal{B}) = p M_n(\mathcal{B})$. It follows that

$$\mathcal{M} = \mathcal{M} M_{1,n}(\mathcal{B}) M_{n,1}(\mathcal{B}) \subseteq p M_n(\mathcal{B}) M_{n,1}(\mathcal{B}) = p M_n(\mathcal{B}).$$

Since $\mathcal{M}$ is closed in the weak operator topology, the reverse inclusion also holds:

$$\mathcal{M} = \mathcal{M} M_{1,n}(\mathcal{B}) M_{n,1}(\mathcal{B}) \supseteq \mathcal{M} M_{1,n}(\mathcal{B}) M_{n,1}(\mathcal{B}) = p M_n(\mathcal{B}) M_{n,1}(\mathcal{B}).$$

Hence $\mathcal{M} = p M_{n,1}(\mathcal{B}) = p\mathcal{B}^n$.

Finally, suppose that $\mathcal{M} \subseteq p\mathcal{B}^n$ for some projection $p \in M_n(\mathcal{B})$, and we must now show that $p \in M_n(\mathcal{C})$. Denote by $\mathcal{B}^{(n)}$ the subalgebra of $M_n(\mathcal{B})$ consisting of all diagonal matrices with the same element along the diagonal. Since $\mathcal{M}$ is a left $\mathcal{B}$-module, we have $\mathcal{B}^{(n)} \mathcal{M} \subseteq \mathcal{M}$ or (equivalently) $(1-p)\mathcal{B}^{(n)} p\mathcal{B}^n = 0$. This implies that $(1-p)\mathcal{B}^{(n)} p = 0$; hence, $p$ commutes with the von Neumann algebra $\mathcal{B}^{(n)}$. It is well known (and easy to verify) that the commutant of $\mathcal{B}^{(n)}$ is $M_n(\mathcal{B}')$ (where $\mathcal{B}'$ is the commutant of $\mathcal{B}$); hence, $p \in M_n(\mathcal{B}) \cap M_n(\mathcal{B}') = M_n(\mathcal{B} \cap \mathcal{B}') = M_n(\mathcal{C})$. □

**Proof of Theorem 2.1.** We may identify the $C^*$-algebra $\mathcal{A}$ with its image under the universal representation on some Hilbert space $\mathcal{H}$. It is well known, then,
that each bounded linear functional on \( \mathcal{A} \) can be uniquely extended to a weak-operator continuous linear functional on the weak-operator closure \( \overline{\mathcal{A}} \) of \( \mathcal{A} \) and, consequently, that \( \overline{\mathcal{A}} = \mathcal{A} \cap \overline{\mathcal{A}} \) for each convex subset \( \mathcal{K} \) of \( \mathcal{A} \) (where one bar denotes the closure in the weak operator topology and two bars denote the norm closure; see \([4, p. 713]\)). It is clear that (if \( n \) is finite) these properties hold also for \( \mathcal{A}^n \) (in place of \( \mathcal{A} \)); in particular, for each \( \tilde{a} \in \mathcal{A}^n \) we have \( \overline{\tilde{a}^*} = \mathcal{A}^n \cap \overline{\tilde{a}} \). Let \( \tilde{b} \in \mathcal{A}^n \) be such that \( \tilde{b} \notin \overline{\tilde{a}} \). Then \( \tilde{b} \notin \overline{\tilde{a}} \). Since \( \overline{\tilde{a}} \) is an \( \mathcal{A} \)-bimodule in \( \mathcal{A}^n \), by Lemma 2.2 there exists a projection \( p \in M_n(\mathcal{C}) \) (where \( \mathcal{C} \) is the centre of \( \overline{\mathcal{A}} \)) such that \( \overline{\tilde{a}} = p\overline{\tilde{a}}^n \). With \( q = 1 - p \), we now have \( q\tilde{a} = 0 \) and \( q\tilde{b} \neq 0 \) (since \( \tilde{b} \notin \overline{\tilde{a}} \)). Thus, if \( c_j \) \((j = 1, \ldots, n)\) are the entries of a suitable row of \( q \), then

\[
\sum_{j=1}^{n} c_j a_j = 0, \quad \sum_{j=1}^{n} c_j b_j \neq 0,
\]

and \( c_j \in \mathcal{C} \) for each \( j \). Now choose any irreducible representation \( \pi \) of \( \overline{\mathcal{A}} \) such that \( \pi(\sum_{j=1}^{n} c_j b_j) \neq 0 \) (this is possible by elementary \( C^* \)-theory \([10, p. 147]\)). Since the centre of each irreducible algebra consists only of scalar multiples of the identity we have \( \pi(c_j) = \lambda_j 1 \), where \( \lambda_j \in \mathcal{C} \). Hence relations (2) imply

\[
\sum_{j=1}^{n} \lambda_j \pi(a_j) = 0 \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j \pi(b_j) \neq 0
\]

or

\[
\sum_{j=1}^{n} \lambda_j a_j \in \ker \pi \cap \mathcal{A} \quad \text{and} \quad \sum_{j=1}^{n} \lambda_j b_j \notin \ker \pi \cap \mathcal{A}.
\]

Thus, with \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \), we have \( \tilde{\lambda} \cdot \tilde{b} \notin [\tilde{\lambda} \cdot \tilde{a}] \). This proves the nontrivial part of the statement in the theorem concerning \( C^* \)-algebras.

The statement concerning von Neumann algebras follows immediately from Lemma 2.2: if \( \tilde{b} \notin \overline{\tilde{a}} \), then there exists a projection \( q \in M_n(\mathcal{C}) \) such that \( q\tilde{a} = 0 \) and \( q\tilde{b} \neq 0 \); hence, for some row \( \tilde{\lambda} \) of \( q \) we have \( \tilde{\lambda} \cdot \tilde{a} = 0 \) and \( \tilde{\lambda} \cdot \tilde{b} \neq 0 \). \( \square \)

The following result is an obvious consequence of Theorem 2.1.

**Corollary 2.3.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( \mathcal{B}(\mathcal{A}) \) the algebra of all bounded linear operators on \( \mathcal{A} \). Then the closure in the strong operator topology (= point-norm topology) of the algebra \( \mathcal{C} \) of all elementary operators on \( \mathcal{A} \) (as a subset of \( \mathcal{B}(\mathcal{A}) \)) consists precisely of operators \( \varphi \in \mathcal{B}(\mathcal{A}) \) that satisfy \( \varphi(\mathcal{K}) \subseteq \mathcal{K} \) for each closed ideal \( \mathcal{K} \) in \( \mathcal{A} \).

If \( \mathcal{A} \) is a von Neumann algebra, then \( \mathcal{C} \) is a dense subset of the space of all bounded module endomorphisms of \( \mathcal{A} \) over the centre \( \mathcal{C} \) equipped with the point-weak operator topology. (Here the point-weak operator topology on \( \mathcal{B}(\mathcal{A}) \) is defined by the family of seminorms \( \psi \rightarrow \|\varphi(a) \xi, \eta\| \), where \( \xi \) and \( \eta \) are arbitrary vectors from the Hilbert space on which \( \mathcal{A} \) is acting.)

An operator \( \varphi \in \mathcal{B}(\mathcal{A}) \) is called a local elementary operator if for each \( x \in \mathcal{A} \) there exists an elementary operator \( E \) (depending on \( x \)) such that

\[
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\]
Larson and Sourour observed in [5] that for each infinite-dimensional Banach space \( \mathcal{X} \) there exist on \( \mathcal{B}(\mathcal{X}) \) nonelementary local elementary operators. Corollary 2.3 implies that on \( C^* \)-algebras local elementary operators can be strongly approximated by elementary operators.

To study the range inclusion problem for elementary operators, we shall need a sharper form of Proposition 1.1 for factors, but to prove it, we need a lemma.

**Lemma 2.4.** Let \( \mathcal{A} \) be a unital prime \( C^* \)-algebra (or, more generally, a unital complex ultraprime Banach algebra in the sense [9]), \( \mathcal{E} \) the algebra of all elementary operators on \( \mathcal{A} \), \( \mathcal{S} \) a finite-dimensional subspace of \( \mathcal{A} \), and \( b \in \mathcal{A} \setminus \mathcal{S} \). Then there exists \( E \in \mathcal{E} \) such that \( E \mathcal{S} = 0 \) and \( Eb \neq 0 \).

**Proof.** The proof is by induction on the dimension \( n \) of \( \mathcal{S} \). First assume that \( n = 1 \), and choose a nonzero element \( a \in \mathcal{S} \). If for each \( E \in \mathcal{E} \) the condition \( Ea = 0 \) implies \( Eb = 0 \), then the correspondence \( Ea \rightarrow Eb \) is a well-defined \( \mathcal{A} \)-bimodule homomorphism from the ideal \( \mathcal{E}a \) to \( \mathcal{A} \), which maps \( a \) to \( b \). By [7, Proposition 2.5] (or, if \( \mathcal{A} \) is a general ultraprime Banach algebra, by [9, Theorem 4.1]) each such homomorphism is necessarily a multiplication by certain \( \gamma \in C \). Hence we have \( b = \gamma a \); but this is in contradiction with \( b \notin \mathcal{S} \). Hence there exists an \( E \in \mathcal{E} \) such that \( E \mathcal{S} = 0 \) and \( Eb \neq 0 \).

Now let \( n \) be any positive integer, assume inductively that the lemma holds for all subspaces of dimension at most \( n \), and let \( \mathcal{S} \) be an arbitrary \((n + 1)\)-dimensional subspace of \( \mathcal{A} \). Choose a basis \( \{a_1, \ldots, a_n, a\} \) for \( \mathcal{S} \) and denote by \( \mathcal{T} \) the span of \( \{a_1, \ldots, a_n\} \). By the inductive hypothesis there exists \( F \in \mathcal{E} \) such that \( F \mathcal{T} = 0 \) and \( Fb \neq 0 \). If \( Fb \notin \mathcal{C}Fa \), then (by the already proved case \( n = 1 \)) there exists \( G \in \mathcal{E} \) such that \( GFa = 0 \) and \( GFb \neq 0 \); hence, \( E \equiv GF \) satisfies \( E \mathcal{S} = 0 \) and \( Eb \neq 0 \). Thus we may assume that \( Fb = \alpha Fa \) for some \( \alpha \in C \). Put \( c = b - \alpha a \) and note that \( Fc = 0 \). Then \( c \notin \mathcal{T} \) (since \( b \notin \mathcal{T} \)). Hence by the inductive hypothesis there exists \( H \in \mathcal{E} \) such that \( H \mathcal{T} = 0 \) and \( Hc \neq 0 \). Since \( \mathcal{A} \) is a prime algebra and \( Fa \neq 0 \), \( Hc \neq 0 \), there exists \( d \in \mathcal{A} \) such that \( F(a)dH(c) 
eq 0 \). Finally, let \( E \in \mathcal{E} \) be defined by

\[
E = -F(x)dH(a) + F(a)dH(x) \quad (x \in \mathcal{A}).
\]

Then \( Ea = 0 \), \( E \mathcal{T} = 0 \) (since \( F \mathcal{T} = 0 \) and \( H \mathcal{T} = 0 \)), and

\[
Eb = E(c + \alpha a) = Ec = -F(c)dH(a) + F(a)dH(c) = F(a)dH(c) \neq 0. \quad \Box
\]

In the proof of our last two results we shall also use the following observation: if \( f_k \ (k = 1, 2, \ldots) \) is an increasing sequence of projections in a factor \( \mathcal{A} \) and \( e \in \mathcal{A} \) is a projection such that \( f_k \ll e \) for each \( k \), then \( f \ll e \), where \( f = \sqrt{\sum_{k=1}^{\infty} f_k} \). If all projections \( f_k \) are finite, this follows from [12, Lemma 2.2, p. 310]. If some \( f_k \) is infinite, then the observation follows from the fact that any two infinite cyclic projections in a factor are equivalent [4, p. 414], by noting that the cardinal number of cyclic summands in \( f \) is less than or equal to the cardinal number of cyclic summands in \( e \) (since \( f = f_1 + \sum_{k=1}^{\infty} (f_{k+1} - f_k) \) and \( f_{k+1} - f_k \ll e \), \( f_1 \ll e \)).

**Proposition 2.5.** Let \( \mathcal{A} \) be a von Neumann factor, \( \mathcal{H} \) a closed ideal in \( \mathcal{A} \), and \( a_1, \ldots, a_n \) elements of \( \mathcal{A} \) linearly independent modulo \( \mathcal{H} \). Then there exist ideals \( \mathcal{I}_j \ (j = 1, \ldots, n) \) in \( \mathcal{A} \) such that \( \mathcal{I}_j \supset \mathcal{H} \) for each \( j \) (where the
symbol $\supset$ is used in the strict sense; that is, $\mathcal{J}_j \neq \mathcal{H}$) and

\[(3)\quad \mathcal{E} \bar{a} \supseteq \bigoplus_{j=1}^{n} \mathcal{J}_j,
\]

where $\bar{a} = (a_1, \ldots, a_n)$.

Proof. The proof is again by an induction on $n$. In the case $n = 1$ it suffices to prove that the ideal $\langle a_1 \rangle$ generated by $a_1$ contains $\mathcal{H}$. Since $a_1 \notin \mathcal{H}$, we see (using the polar decomposition of $a_1$ and the spectral theorem) that the spectral projection $e$ of $|a_1|$ corresponding to a certain positive interval satisfies $e \notin \mathcal{H}$ and $e \in \langle a_1 \rangle$. Hence it suffices to prove that $\mathcal{H} \subseteq \langle e \rangle$. For this, it suffices to show that $b \in \langle e \rangle$ for each positive $b$ in $\mathcal{H}$. For each $k = 1, 2, \ldots$, let $f_k$ be the spectral projection of $b$ corresponding to the interval $[1/k, \|b\|]$. Then $f \equiv \bigwedge_{k=1}^{\infty} f_k$ is the range projection of $b$; hence, $b = fb$. From $f_k \in \mathcal{H}$, $e \notin \mathcal{H}$, and the fact that any two projections in a factor are comparable [4, p. 408], it follows that $f_k \prec e$ for each $k$; hence, $f \preceq e$. This implies that $f \in \langle e \rangle$; therefore, we have $b = fb \in \langle e \rangle$.

Now let $n > 1$. If we can prove that there exists an $E \in \mathcal{E}$ such that

\[(4)\quad Ea_j = 0 \text{ for } j = 1, \ldots, n-1\quad \text{and} \quad Ea_n \notin \mathcal{H},
\]

then by a previous paragraph (applied to $Ea_n$) we have

\[\mathcal{E} \bar{a} \supseteq \mathcal{E}(0, \ldots, 0, Ea_n) = 0 \oplus \cdots \oplus 0 \oplus \langle Ea_n \rangle \supseteq 0 \oplus \cdots \oplus 0 \oplus \mathcal{H},
\]

and by applying the same arguments also to other components, the proposition follows. To prove (4), assume, to the contrary, that the condition $Ea_j = 0$ for $j = 1, \ldots, n-1$ implies that $Ea_n \in \mathcal{H}$. Then the map

\[\varphi: \mathcal{E}(a_1, \ldots, a_{n-1}) \to \mathcal{A}/\mathcal{H},
\]

\[\varphi(Ea_1, \ldots, Ea_{n-1}) \equiv Ea_n + \mathcal{H} \quad (E \in \mathcal{E})
\]

is a well-defined homomorphism of $\mathcal{E}$-modules. By the inductive hypothesis we have $\mathcal{E}(a_1, \ldots, a_{n-1}) \supseteq \mathcal{H}^{n-1}$. The components $\varphi_j$ of the restriction of $\varphi$ to $\mathcal{H}^{n-1}$ are $\mathcal{E}$-module homomorphisms from $\mathcal{H}$ to $\mathcal{A}/\mathcal{H}$; hence, $\varphi_j = 0$ for each $j = 1, \ldots, n-1$ (since $\varphi_j(\mathcal{H}) = \varphi_j(\mathcal{H}) = \mathcal{H} \varphi_j(\mathcal{H}) = 0$ in $\mathcal{A}/\mathcal{H}$). It follows that $\varphi(\mathcal{H}^{n-1}) = 0$; hence, $\varphi$ induces an $\mathcal{E}$-module homomorphism $\tilde{\varphi}: \mathcal{E}(a_1, \ldots, a_{n-1})/\mathcal{H}^{n-1} \to \mathcal{A}/\mathcal{H}$. Identifying in the obvious way $\mathcal{E}(a_1, \ldots, a_{n-1})/\mathcal{H}^{n-1}$ with $\tilde{\mathcal{E}}(a_1 + \mathcal{H}, \ldots, a_{n-1} + \mathcal{H})$, where $\tilde{\mathcal{E}}$ denotes the algebra of all elementary operators on $\mathcal{A}/\mathcal{H}$, we can regard $\tilde{\varphi}$ as an $\tilde{\mathcal{E}}$-module homomorphism from $\tilde{\mathcal{E}}(a_1 + \mathcal{H}, \ldots, a_{n-1} + \mathcal{H})$ to $\mathcal{A}/\mathcal{H}$ such that $\tilde{\varphi}(E(a_1 + \mathcal{H}), \ldots, E(a_{n-1} + \mathcal{H})) = \tilde{E}(a_n + \mathcal{H}) \ (E \in \tilde{\mathcal{E}})$. In particular, for any fixed $\tilde{E} \in \tilde{\mathcal{E}}$ we have $\tilde{E}(a_n + \mathcal{H}) = 0$ if $\tilde{E}(a_j + \mathcal{H}) = 0$ for all $j = 1, \ldots, n-1$. Hence by Lemma 2.4 $a_n + \mathcal{H}$ must be contained in the subspace of $\mathcal{A}/\mathcal{H}$ spanned by $\{a_j + \mathcal{H}: j = 1, \ldots, n-1\}$, but this is in contradiction with the assumed linear independence modulo $\mathcal{H}$ of elements $a_j$ ($j = 1, \ldots, n$). (Here we have used the fact that $\mathcal{A}/\mathcal{H}$ is a prime $C^*$-algebra, since the closed ideals in a factor are linearly ordered by inclusion [4, p. 451].) □
Proposition 2.5 can be useful in studying the question of when the range of a fixed elementary operator is contained in a given (not necessarily closed) ideal \( \mathcal{I} \) of \( \mathcal{A} \). In the special case \( \mathcal{I} = 0 \), Mathieu \([7, 9]\) solved the problem for general prime \( \mathcal{C}^* \)-algebras and ultraprime Banach algebras. In the case \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \) the question was studied in \([3, 1, 2, 6]\); in particular, the case \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \) of the following corollary was proved by Apostol and Fialkow \([1, \text{Theorem 3.1}]\).

**Corollary 2.6.** Let \( \mathcal{I} \) be a (not necessarily closed) ideal in a factor \( \mathcal{A} \), and let \( E \) be an elementary operator on \( \mathcal{A} \) defined by

\[
E_x = \sum_{j=1}^{n} a_j x b_j \quad (x \in \mathcal{A}),
\]

where \( \tilde{a} = (a_1, \ldots, a_n) \) and \( \tilde{b} = (b_1, \ldots, b_n) \) are fixed elements in \( \mathcal{A}^n \). If \( E(\mathcal{A}) \subseteq \mathcal{I} \) and \( a_1, \ldots, a_n \) are linearly independent modulo the norm closure \( \overline{\mathcal{I}} \) of \( \mathcal{I} \) then each \( b_j \) (\( j = 1, \ldots, n \)) must be in \( \mathcal{I} \).

**Proof.** We shall prove that \( b_1 \in \mathcal{I} \); the proof that \( b_j \in \mathcal{I} \) for \( j > 1 \) is the same. By Proposition 2.5 there exists an elementary operator \( F \), say

\[
F_x = \sum_{i=1}^{m} u_i x v_i \quad (x \in \mathcal{A}),
\]

where \( u_j, v_j \in \mathcal{A} \) are fixed, such that

\[
F_a_j = 0 \quad \text{for} \quad j = 2, \ldots, n \quad \text{and} \quad F a_1 \notin \overline{\mathcal{I}}.
\]

Since by hypothesis \( E(\mathcal{A}) \subseteq \mathcal{I} \) and \( \mathcal{I} \) is an ideal, we have \( \sum_{i=1}^{m} u_i E(v_i x) \in \mathcal{I} \) for all \( x \in \mathcal{A} \), which can be written as

\[
\sum_{i=1}^{m} u_i \sum_{j=1}^{n} a_j v_i x b_j \in \mathcal{I} \quad (x \in \mathcal{A}).
\]

Reversing the order of summation and using (5) we obtain that \( (F a_1) x b_1 \in \mathcal{I} \) for all \( x \in \mathcal{A} \). Denoting \( F a_1 \) by \( a \) and \( b_1 \) by \( b \), we now have

\[
a \mathcal{A} b \subseteq \mathcal{I} \quad \text{and} \quad a \notin \overline{\mathcal{I}},
\]

and we must prove that this implies that \( b \in \mathcal{I} \).

Using the polar decomposition of \( a \) and \( b^* \) and the spectral theorem for \( |a| = \sqrt{a^* a} \) it follows from (6) by classical arguments (see \([4, \S 6.8]\)) that for some spectral projection \( e \) of \( |a| \) we have

\[
e \mathcal{A} |b^*| \subseteq \mathcal{I} \quad \text{and} \quad e \notin \overline{\mathcal{I}}.
\]

For each \( k = 1, 2, \ldots \) let \( f_k \) be the spectral projection of \( |b^*| \) corresponding to \( (1/k, \infty) \). Since any two projections in a factor are comparable \([4, \text{p. 408}]\), there are now two cases: (i) \( e \leq f_k \) for some positive integer \( k \); (ii) \( f_k \prec e \) for each \( k \); but, from \( e \mathcal{A} |b^*| \subseteq \mathcal{I} \) we have \( e \mathcal{A} f_k \subseteq \mathcal{I} \) for each \( k \). Hence in the first case it follows that \( e \in \mathcal{I} \) (since \( e = e u f_k u^* \in \mathcal{I} u^* \subseteq \mathcal{I} \), where \( u \in \mathcal{A} \) is a partial isometry with initial projection contained in \( f_k \) and final projection \( e \)), which is in contradiction with \( e \notin \overline{\mathcal{I}} \). Thus, only case (ii) occurs. Since
for all $k$, the range projection $f = \bigvee_{k=1}^{\infty} f_k$ of $|b^*|$ satisfies $f \preceq e$. Denoting by $v$ a partial isometry with initial projection $f$ and final projection contained in $e$, we have $|b^*| = f|b^*| = v^*ev|b^*| \in \mathcal{S}$ (since $e \mathcal{S} |b^*| \subseteq \mathcal{S}$). Hence $b \in \mathcal{S}$ by polar decomposition. 

**References**


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