A VECTORIAL SLEPIAN TYPE INEQUALITY. APPLICATIONS

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Abstract. We prove a new inequality for Gaussian processes; this inequality implies the Chevet's inequality and Gordon's inequalities. Some remarks on Gaussian proofs of Dvoretzky's theorem are given.

I. Introduction

Let \( \{g_{i,k}\} (1 \leq i \leq n, 1 \leq k \leq d), \{h_k\}_1^d, \) and \( \{g_i\}_1^n \) denote independent sets of orthonormal Gaussian random variables. Let \( E \) and \( F \) be Banach spaces, \( \{f_k\}_{k=1}^d \subset F \) and \( \{x_i^*\}_{i=1}^n \subset E^* \). Let \( T(\omega) = \sum_{i=1}^n \sum_{k=1}^d g_{i,k}(\omega)x_i^* \otimes f_k \) be a random operator from \( E \) to \( F \). The Chevet inequality says \([Cv]\)

\[
\mathbb{E} \left( \max_{\|x\|_E = 1} \|T_\omega x\| \right) \leq \sqrt{2} \left( \varepsilon_2(x_1^*, \ldots, x_n^*) \mathbb{E} \left( \left\| \sum_{k=1}^d h_k f_k \right\| \right) + \varepsilon_2(f_1, \ldots, f_d) \mathbb{E} \left( \left\| \sum_{i=1}^n g_i x_i^* \right\|_{E^*} \right) \right),
\]

where

\[\varepsilon_2(x_1^*, \ldots, x_n^*) = \sup \left\{ \left( \sum_{1 \leq i \leq n} x_i^* (x)^2 \right)^{1/2} \right\}, \|x\|_E \leq 1\]

and

\[\varepsilon_2(f_1, \ldots, f_d) = \sup \left\{ \left( \sum_{1 \leq k \leq d} y^* (f_k)^2 \right)^{1/2} \right\}, \|y^*\|_{E^*} \leq 1\].

Later, Gordon proved an inequality in the opposite direction:

\[
\inf_{\|x\|_E = 1} \left\{ \left( \sum_{i=1}^n x_i^* (x)^2 \right)^{1/2} \right\} \mathbb{E} \left( \left\| \sum_{k=1}^d h_k f_k \right\| \right) - \varepsilon_2(f_1, \ldots, f_d) \mathbb{E} \left( \left\| \sum_{i=1}^n g_i x_i^* \right\|_{E^*} \right) \leq \mathbb{E} \left( \min_{\|x\|_E = 1} \|T_\omega x\| \right).
\]

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He also showed that the constant \( \sqrt{2} \) in (1.1) can be replaced by 1 (see [G1]). Our aim is to deduce these inequalities from a general Gaussian inequality for Gaussian processes.

II. Basic inequalities

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space and \(X\) a canonical \(\mathbb{R}^d\)-valued Gaussian random vector (i.e., with covariance matrix equal to \(\text{Id}_d\)). We define two Gaussian processes as follows. For \(n \geq 1\), let \(B^2_n\) be the closed unit ball of \(l^2_n\) and \(S^{n-1}\) its unit sphere. For \(x = (x^1, \ldots, x^n) \in \mathbb{R}^n\), let \(\|x\|_2 = (\sum_{i=1}^n (x^i)^2)^{1/2}\) and let \(X_1, \ldots, X_n\) be \(n\) independent copies of \(X\), independent of \(X\). Let \(\{g_1, \ldots, g_n\}\) be a set of orthonormal Gaussian random variables independent of \(\{X, X_1, \ldots, X_n\}\). Let

\[
X_x = \sum_{i=1}^n x^i X_i \quad \text{and} \quad g_x = \sum_{i=1}^n x^i g_i.
\]

We shall prove the following inequality.

**Theorem 2.1.** Let \(A \subseteq B^2_n\). Let \(F_x : \mathbb{R}^d \to \mathbb{R}\) be a family of 1-Lipschitz functions indexed by \(x \in A\). Then the Gaussian processes \(\{X_x\}_{x \in A}\) and \(\{g_x\}_{x \in A}\) satisfy

\[
\mathbb{E} \max_{x \in A} F_x(X_x) \leq \mathbb{E} \max_{x \in A} \{F_x(\|x\|_2 X) + g_x\}.
\]

**Corollary 2.1.** Let \(A \subseteq B^2_n\), and let \(\|\cdot\|\) be a norm on \(\mathbb{R}^d\) such that \(\forall x \in \mathbb{R}^d\), \(\|\cdot\| \leq \|x\|_2\). Then the processes \(\{X_x\}_{x \in A}\) and \(\{g_x\}_{x \in A}\) verify

\[
\min_{x \in A} \|x\|_2 \|X_x\| - \mathbb{E} \max_{x \in A} g_x \leq \mathbb{E} \min_{x \in A} \|X_x\| \leq \mathbb{E} \max_{x \in A} \|X_x\| \leq \mathbb{E} \|X\| + \mathbb{E} \max_{x \in A} g_x.
\]

**Proof.** For the right-hand side inequality put \(F_y(x) = \|x\|\), and for the left-hand side inequality put \(F_y(x) = -\|x\|\). \(\Box\)

**Corollary 2.2.** Let \(X\) be a canonical \(\mathbb{R}^d\)-valued Gaussian random vector, with \(X_x\) and \(g_x\) as defined in (2.1). Let \(A \subseteq S^{n-1}\), \(F\) a 1-Lipschitz function on \(\mathbb{R}^d\), and \(\mu = \mathbb{E} F(X)\). Then the processes \(\{X_x\}_{x \in A}\) and \(\{g_x\}_{x \in A}\) verify

\[
\mathbb{E} \max_{x \in A} |F(X_x) - \mu| \leq \mathbb{E} |F(X) - \mu| + \mathbb{E} \max_{x \in A} g_x \leq 1 + \mathbb{E} \max_{x \in A} g_x.
\]

**Proof.** For the first inequality, take \(G(\cdot) = |F(\cdot) - \mu|\), which is a 1-Lipschitz function; for the second, we use a well-known Poincaré-type inequality, that is,

\[
\mathbb{E} |f(X) - \mathbb{E}(f(X))|^2 \leq \mathbb{E} \|\nabla f(X)\|_2^2
\]

for \(X\) as above and all 1-Lipschitz functions \(f\) on \(\mathbb{R}^d\) [P, C]. \(\Box\)

Next we show how the Gordon inequalities follow from inequality (2.3). Indeed, let \(u : \mathbb{R}^d \to F\), \(u(\sum_{k=1}^d \alpha^k e_k) = \sum_{k=1}^d \alpha^k f_k\), and \(v : E \to l^2_n\), \(v(x) = (x^1_1(x), \ldots, x^*_{n}(x))\). We have \(\|u\| = e_2(f_1, \ldots, f_d)\) and \(\|v\| = e_2(x^1_1, \ldots, x^*_{n})\). Let \(X = \sum_{k=1}^d h_k e_k\), and for \(1 \leq i \leq n\) let \(X_i = \sum_{k=1}^d g_{ik} e_k\). Then \(X\) is an \(\mathbb{R}^d\)-valued canonical Gaussian vector and \(X_1, \ldots, X_n\) are \(n\) independent copies of \(X\), independent of \(X\). Then \(u(X_u(x)(\omega)) = T_{\omega}(x)\), so the rest of the proof is as in Corollary 2.1 with \(A = v(S_E)\), where \(S_E\) is the unit sphere of \(E\) and \(\|\alpha\| = \|u(\alpha)\|\). \(\Box\)
Before proving Theorem 2.1, we get a vectorial Slepian type inequality, from which we deduce Theorem 2.1 (see Theorem 2.2).

We define some notation. For $x = (x_i), y = (y_i)$ in $\mathbb{R}^d$, $x \otimes y$ denotes the matrix $(x_i y_j)_{1 \leq i, j \leq d}$, and for $u, v \in \mathbb{R}^d$, define $x \otimes y[u, v]$ as $\langle u, x \otimes y(v) \rangle = \langle x, u \rangle \langle y, v \rangle$ and $\| \cdot \|_{\mathcal{S}(\mathbb{R}^d)}$ as the operator norm.

**Theorem 2.2.** Let $\{X_t\}$ and $\{Y_t\}$, $t \in T$, be two families of Gaussian vectors with values in $\mathbb{R}^d$, let $\{g_t\}$ be a family of Gaussian random variables independent of $\{X_t\}$ and $\{Y_t\}$, and suppose

1. $\text{dist}(X_t) = \text{dist}(Y_t)$ for all $t \in T$,
2. $\|\mathbb{E}(X_t \otimes X_s - Y_t \otimes Y_s)\|_{\mathcal{S}(\mathbb{R}^d)} \leq \frac{1}{2} \mathbb{E}|g_t - g_s|^2$ for all $s, t$ in $T$.

Let $F_t, t \in T$, be a family of real 1-Lipschitz functions on $\mathbb{R}^d$. Then

$$\mathbb{E} \sup_t F_t(X_t) \leq \mathbb{E} \sup_t \{F_t(Y_t) + g_t\}.$$

**Proof.** We may clearly assume without loss of generality that the two processes $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ are independent and, also by a standard approximation argument, that the $F_t$ are 1-Lipschitz and twice differentiable.

It is clear that we just need to prove the inequality for finite sets $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ ($N \geq 1$). Fix $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$, and prove that

$$(2.4) \quad \mathbb{E} \max_{1 \leq i \leq N} \{F_i(X_i)\} \leq \mathbb{E} \max_{1 \leq i \leq N} \{F_i(Y_i) + g_i\}.$$ 

For $\theta \in [0, \pi/2]$ let

$$Z(\theta) = (\cos(\theta)X_1 + \sin(\theta)Y_1, \sin(\theta)g_1; \ldots; \cos(\theta)X_N + \sin(\theta)Y_N, \sin(\theta)g_N)$$

where $Z(\theta)$ is an $(\mathbb{R}^{d+1})^N$-valued Gaussian vector, with

$$Z(0) = (X_1, 0; \ldots; X_N, 0) \quad \text{and} \quad Z(\pi/2) = (Y_1, g_1; \ldots; Y_N, g_N);$$

a vector $(y, z)$ of $E = (\mathbb{R}^{d+1})^N$ will be denoted by

$$(y, z) = ((y_i, z_i))_{1 \leq i \leq N} \quad \text{where} \quad y_i \in \mathbb{R}^d \text{ and } z_i \in \mathbb{R}.$$ 

We prove first the following lemma.

**Lemma 2.1.** Let $F: (\mathbb{R}^{d+1})^N \to \mathbb{R}^N$, $F(y, z) = (F_1(y_1) + z_1, \ldots, F_N(y_N) + z_N)$ where $F_1, \ldots, F_N$ are 1-Lipschitz twice differentiable on $\mathbb{R}^d$, and $G: \mathbb{R}^N \to \mathbb{R}$ be a twice differentiable function such that $\exists k_1, k_2$, such that $|G(\cdot)| \leq k_1 e^{k_2 \|\cdot\|_{l^2}}$, $|\partial_G(\cdot)/\partial \alpha_i| \leq k_1 e^{k_2 \|\cdot\|_{l^2}}$, and $|\partial^2 G(\cdot)/\partial \alpha_i \partial \alpha_j| \leq k_1 e^{k_2 \|\cdot\|_{l^2}}$ for all $i, j = 1, \ldots, N$. Put $\varphi = G \circ F$ and

$$(2.5) \quad h(\theta) = \mathbb{E}\varphi(Z(\theta)).$$

Suppose

$$(2.6) \quad \forall i, j, i \neq j, \quad \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \leq 0$$

and

$$(2.7) \quad \forall j = 1, \ldots, N \quad \sum_{i=1}^N \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} = 0.$$
Then \( h(\theta) \) is increasing; therefore,
\[
\mathbb{E}G(F_1(X_1), \ldots, F_N(X_N)) = h(0) \leq h(\pi/2) = \mathbb{E}G(F_1(Y_1) + g_1, \ldots, F_N(Y_N) + g_N).
\]

Proof of Lemma 2.1. Let \( \epsilon > 0 \), and let \( \Lambda \) be an \((\mathbb{R}^{d+1})^N\)-valued canonical Gaussian vector independent of \( \{Z(\theta); \theta \in [0, \pi/2]\} \). Let \( Z_\epsilon(\theta) = Z(\theta) + \epsilon\Lambda \) so that \( \Gamma_\epsilon(\theta) = \Gamma(\theta) + \epsilon^2 I_E \), where \( \Gamma(\theta) \) is the covariance matrix of \( Z(\theta) \) and \( \Gamma_\epsilon(\theta) \) is the covariance matrix of \( Z_\epsilon(\theta) \). Thus
\[
\Gamma_\epsilon(\theta) \to \Gamma(\theta) \text{ as } \epsilon \to 0 \quad \text{so that} \quad h_\epsilon(\theta) \to h(\theta) \text{ as } \epsilon \to 0.
\]

Remark that
\[
\forall (u, v) \in E \quad \langle (u, v), \Gamma_\epsilon(\theta)(u, v) \rangle \geq \epsilon^2 \| (u, v) \|_E^2.
\]

Let \( g_\epsilon(y, z; \theta) \) be the density function of \( Z_\epsilon(\theta) \). We will list the following well-known identities (see \([G2, F, G1]\)):
\[
(2.8) \quad g_\epsilon(y, z; \theta) = \frac{1}{(2\pi)^{(d+1)N}} \int_E \exp \left\{ \frac{1}{2} \langle (u, v), \Gamma_\epsilon(\theta)(u, v) \rangle - \frac{1}{2} \langle (u, v), \Gamma(\theta)(u, v) \rangle \right\} dudv
\]
where \( du = du_1 \cdots du_N, \; du_i = du_{i,1} \cdots du_{i,d}, \) and \( dv = dv_1 \cdots dv_N; \)
\[
(2.9) \quad h_\epsilon(\theta) = \int_E \varphi(y, z)g_\epsilon(y, z, \theta) dy dz = \mathbb{E}\varphi(Z_\epsilon(\theta));
\]
\[
(2.10) \quad h_\epsilon'(\theta) = \int_E \varphi(y, z) \frac{\partial}{\partial \theta} g_\epsilon(y, z, \theta) dy dz;
\]
\[
(2.11) \quad \frac{\partial}{\partial \theta} g_\epsilon(x, \theta) = \frac{1}{2} \sum_{i,j=1}^{(d+1)N} \frac{d}{d\theta} \gamma_{i,j}(\theta) \frac{\partial^2}{\partial x_i \partial x_j} g_\epsilon(x, \theta)
\]
where \( x = (y, z) \) and \( \Gamma_\epsilon(\theta) = (\gamma_{i,j}(\theta))_{1 \leq i,j \leq N(d+1)} \). We compute \( \Gamma_\epsilon(\theta) \). We can write \( \Gamma_\epsilon(\theta) \) as a block matrix:
\[
\Gamma_\epsilon(\theta) = (\Gamma_{i,j}(\theta))_{1 \leq i \leq N, \; 1 \leq j \leq N}
\]
where
\[
(2.12) \quad \Gamma_{i,j}(\theta) = \mathbb{E}[Z_i^\epsilon(\theta) \otimes Z_j^\epsilon(\theta)]
\]
where
\[
Z_i^\epsilon(\theta) = (X_i(\theta) + Y_i(\theta) + \epsilon \Lambda_{i}, g_i(\theta) + \epsilon \Lambda_i^\prime)
\]
where
\[
\Lambda = (\Lambda_i, \Lambda_i^\prime)_{1 \leq i \leq N}, \quad \Lambda_i = (\Lambda_i^1, \ldots, \Lambda_i^d),
\]
\[
X_i(\theta) = \cos(\theta)X_i, \quad Y_i(\theta) = \sin(\theta)Y_i, \quad g_i(\theta) = \sin(\theta)g_i.
\]
Using the fact that \( \{X_1, \ldots, X_N\}, \; \{Y_1, \ldots, Y_N\}, \) and \( \{g_1, \ldots, g_N\} \) are independent processes, we find that
\[
(2.13) \quad \Gamma_{i,j}(\theta) = \begin{bmatrix}
A_{i,j}(\theta) + \epsilon^2 I_d & 0 \\
0 & B_{i,j}(\theta)
\end{bmatrix}
\]
where \( A_{i,j}(\theta) \) is a \( d \times d \) matrix and \( B_{i,j}(\theta) \) is a scalar such that
\[
A_{ij}(\theta) = \cos^2(\theta)\mathbb{E}(X_i \otimes X_j) + \sin^2(\theta)\mathbb{E}(Y_i \otimes Y_j),
\]
\[
B_{i,j}(\theta) = \sin^2(\theta)\mathbb{E}g_i g_j + \epsilon^2 \delta_{i,j}
\]
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where \( \delta_{i,j} = 1 \) if \( i = j \), and 0 if \( i \neq j \). A simple computation gives

\[
\langle (u, v); \Gamma_\varepsilon(\theta)(u, v) \rangle
= \sum_{i=1}^{N} \sum_{j=1}^{N} [(u_i, A_{i,j}(\theta)u_j) + \varepsilon^2(u_i, u_j)] + \sum_{i=1}^{N} \sum_{j=1}^{N} B_{i,j}^\varepsilon(\theta)v_i \cdot v_j.
\]

Considering \( \partial^2 g_\varepsilon(y, z; \theta) / \partial y_i \partial y_j \) as a \( d \times d \) matrix for each \( i, j \) gives

\[
\frac{\partial}{\partial \theta} g_\varepsilon(y, z; \theta) = \frac{1}{2} \sum_{i,j=1}^{N} \text{trace} \left( \frac{\partial^2}{\partial y_i \partial y_j} g_\varepsilon(y, z; \theta) \frac{d}{d \theta} A_{i,j}(\theta) \right)
+ \frac{1}{2} \sum_{i,j=1}^{N} \frac{d}{d \theta} B_{i,j}^\varepsilon(\theta) \frac{\partial^2}{\partial z_i \partial z_j} g_\varepsilon(y, z; \theta);
\]

but

\[
(2.15) \quad h_\varepsilon'(\theta) = \int \varphi(y, z) \frac{\partial}{\partial \theta} g_\varepsilon(y, z, \theta) \, dy \, dz.
\]

Let \( M_{i,j} = \mathbb{E} Y_i \otimes Y_j - \mathbb{E} X_i \otimes X_j \). We get

\[
(2.16) \quad h_\varepsilon'(\theta) = \frac{\sin 2\theta}{2} \int \left\{ \sum_{i,j=1}^{N} \text{trace} \left( \frac{\partial^2 \varphi(y, z)}{\partial y_i \partial y_j} \cdot M_{i,j} \right) \right. \\
+ \sum_{i,j=1}^{N} \frac{\partial^2 \varphi(y, z)}{\partial z_i \partial z_j} \mathbb{E} g_i g_j \left. \right\} g_\varepsilon(y, z; \theta) \, dy \, dz.
\]

Since \( \text{dist}(X_i) = \text{dist}(Y_i) \) for all \( i \), we get \( M_{i,i} = 0 \); hence, we have, for \( \varphi = G \circ F \),

\[
(2.17) \quad h_\varepsilon'(\theta) = \frac{\sin 2\theta}{2} \int \left\{ \sum_{i \neq j}^{N} \text{tr} \left( \frac{\partial^2 G \circ F}{\partial y_i \partial y_j} \cdot M_{i,j} \right) \right. \\
+ \sum_{i,j=1}^{N} \left( \frac{\partial^2 G \circ F}{\partial z_i \partial z_j} \right) \mathbb{E} g_i g_j \left. \right\} g_\varepsilon(y, z; \theta) \, dy \, dz.
\]

A simple computation gives, for all \( i \neq j \),

\[
\frac{\partial^2 G \circ F}{\partial y_i \partial y_j} = \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \circ F \cdot \nabla F_i(y_i) \otimes \nabla F_j(y_j)
\]

and

\[
\frac{\partial^2 G \circ F}{\partial z_i \partial z_j} = \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \circ F \quad \text{for all } i, j.
\]

Condition (2.7) gives

\[
(2.17) \quad \frac{\partial^2 G}{\partial \alpha_i^2} = - \sum_{j=1,j \neq i}^{N} \frac{\partial^2 G}{\partial \alpha_i \partial \alpha_j} \quad \text{for all } i, j.
\]
so

\[ h'_\varepsilon(\theta) = \frac{\sin 2\theta}{2} \int \left\{ \sum_{i \neq j} \text{tr} \left( \frac{\partial^2 G(F(y, z))}{\partial y_i \partial y_j} \cdot M_{i,j} \right) + \sum_{i \neq j} \frac{\partial^2 G(F(y, z))}{\partial z_i \partial z_j} \right\} g_\varepsilon(y, z; \theta) \, dy \, dz \]

\[ + \sum_{i=1}^N \frac{\partial^2 G(F(y, z))}{\partial z_i^2} \mathbb{E}g_i^2 \left\} g_\varepsilon(y, z; \theta) \, dy \, dz \]

\[ = \frac{\sin 2\theta}{2} \int \left\{ \sum_{i \neq j} \left( \frac{\partial^2 G(F(y, z))}{\partial y_i \partial y_j} \cdot M_{i,j} \right) + \left( \mathbb{E}g_i g_j - \frac{1}{2} \mathbb{E}g_i^2 + \mathbb{E}g_j^2 \right) \times \frac{\partial^2 G(F(y, z))}{\partial \alpha_i \partial \alpha_j} \right\} g_\varepsilon(y, z; \theta) \, dy \, dz \]

Since \( \|\nabla F_i(y_i)\| \leq 1 \),

\[ \langle M_{i,j}(\nabla F_i(y_i)), \nabla F_j(y_j) \rangle - \frac{1}{2} \mathbb{E}|g_i - g_j|^2 \leq \|(M_{i,j})\|_{\mathcal{L}(\mathbb{R}^d)} - \frac{1}{2} \mathbb{E}|g_i - g_j|^2 \leq 0, \]

so \( h'_\varepsilon(\theta) \geq 0 \) and \( \mathbb{E}G(F(Z_\varepsilon(0))) \leq \mathbb{E}G(F(Z_\varepsilon(\pi/2))) \). Finally, letting \( \varepsilon \to 0 \), we get the result of Lemma 2.1. □

We now finish the proof of Theorem 2.2. The map max which assigns to each \((\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N\) the value \( \max(\alpha_1, \ldots, \alpha_N) \) is slowly increasing and verifies (2.6) and (2.7) in distribution sense [G2]. So if we regularise max by convolution with a twice differentiable function \( \psi_k \), which is supported by a ball of radius \( 1/k \), we obtain a function \( m_k \), which is \( 1 \)-Lipschitz and satisfies the above three conditions. By considering the functions \( h_k(\theta) = \mathbb{E}m_k \circ F(Z(\theta)) \), and by letting \( k \) go to infinity, we find by Lebesgue’s theorem that the function \( \mathbb{E}\max \circ F(Z(\cdot)) \) is increasing in \([0; \pi/2]\). This completes the proof of Theorem 2.2. □

Proof of Theorem 2.1. We have \( X_x = \sum_{i=1}^n x^i X_i \). Let \( Y_x = \|x\|_2 X \), where \( x \) runs over a set \( A \subset B^2_n \). Then \( \text{dist}(X_x) = \text{dist}(Y_x) \). Take a finite set \( \{a_1, \ldots, a_N\} \) in \( A \); a simple computation gives

\[ M_{i,j} = \mathbb{E}(Y_{a_i} \otimes Y_{a_j} - X_{a_i} \otimes X_{a_j}) = (\|a_i\|_2 a_j) \mathbb{E} \text{Id}_d \]

where \( a_i \cdot a_j \) is the scalar product. Moreover, \( \mathbb{E}|g_{a_i} - g_{a_j}|^2 = \|a_i - a_j\|_2^2 \), the \( F_i \) are \( 1 \)-Lipschitz functions, so

\[ \|M_{i,j}\|_{\mathcal{L}(\mathbb{R}^d)} - \frac{1}{2} \mathbb{E}|g_{a_i} - g_{a_j}|^2 = (\|a_i\|_2 a_j) - \frac{1}{2} \|a_i - a_j\|_2^2 \]

\[ = -\frac{1}{2} \|a_i^2 - a_j^2\|_2^2 \leq 0. \]
Hence conditions (i) and (ii) of Theorem 2.2 are satisfied, and Theorem 2.1 is proved. □

III. Final remarks

We give now a short proof of a result due to Milman.

**Theorem 3.1 [M, Sc].** Let $e > 0$, $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function with constant $L$, $X = \sum_{i=1}^{N} g_i e_i$ where $\{g_i\}_{1 \leq i \leq N}$ is a set of orthonormal Gaussian random variables and $\{e_i\}_{1 \leq i \leq N}$ is the canonical basis of $l_2^N$, and $\mu = \mathbb{E} f(X)$. Then there exists an operator $T : l_2^N \to \mathbb{R}^N$ with $n = \lceil (e\mu / L)(e\mu / L - 2) \rceil$, such that

$$|f(Tx) - \mu| \leq e\mu \quad \text{for all } x \in S^{n-1}.$$

**Proof.** Consider, as above, real-valued Gaussian operator $T_\omega = \sum_{i=1}^{N} \sum_{j=1}^{N} g_i e_i \otimes e_j$ from $l_2^N$ to $\mathbb{R}^N$,

$$X_i = \sum_{j=1}^{N} g_{i,j} e_j \quad \text{and} \quad X_x = \sum_{i=1}^{n} x^i X_i$$

where $x = (x^1, \ldots, x^n)$. Then $X_x(\omega) = T_\omega x$, and we have

$$\mathbb{P}(\{\omega / \exists x \in S^{n-1}; |f(X_x) - \mu| > e\mu\}) = \mathbb{P} \left( \left\{ \omega; \sup_{x \in S^{n-1}} |f(X_x) - \mu| > e\mu \right\} \right) \leq \frac{1}{e\mu} \mathbb{E} \sup_{x \in S^{n-1}} |f(X_x) - \mu|.$$

We apply Corollary 2 to get

$$\mathbb{P}(\{\omega / \exists x \in S^{n-1}; |f(X_x) - \mu| > e\mu\}) \leq \frac{1}{e\mu} \left\{ \mathbb{E} |f(X) - \mu| + L \mathbb{E} \sup_{x \in S^{n-1}} \sum_{j=1}^{n} x^j g_j \right\},$$

and using the Poincaré-type inequality as in Corollary 2, we find that

$$\mathbb{P}(\{\omega / \exists x \in S^{n-1}; |f(X_x) - \mu| > e\mu\}) \leq \frac{L}{e\mu} \left[ 1 + \mathbb{E} \sup_{x \in S^{n-1}} \sum_{j=1}^{n} x^j g_j \right] \leq \frac{L}{e\mu} (1 + \sqrt{n}).$$

We only need to choose $n$ such that this last expression is $< 1$.

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**References**


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