ON APPROXIMATELY CONVEX FUNCTIONS

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ABSTRACT. The Bernstein-Doetsch theorem on midconvex functions is extended to approximately midconvex functions and to approximately Wright convex functions.

Let $X$ be a real vector space, $D$ be a convex subset of $X$, and $\varepsilon$ be a nonnegative constant. A function $f: D \to \mathbb{R}$ is said to be

- $\varepsilon$-convex if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$ (cf. [2]);
- $\varepsilon$-Wright-convex if $f(tx + (1-t)y) + f((1-t)x + ty) \leq f(x) + f(y) + 2\varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$;
- $\varepsilon$-midconvex if $f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)) + \varepsilon$ for all $x, y \in D$.

Notice that $\varepsilon$-convexity implies $\varepsilon$-Wright-convexity, which in turn implies $\varepsilon$-midconvexity, but not the converse. The usual notions of convexity, Wright-convexity, and midconvexity correspond to the case $\varepsilon = 0$. A comprehensive review on this subject can be found in [1, 6, 8-10]. The Bernstein-Doetsch theorem relates local boundedness, midconvexity, and convexity (cf. [6, 10]). In order to extend this result to approximately midconvex functions, we first specify the assumptions on the topology $\mathcal{T}$ to be imposed on $X$: the map $(t, x, y) \to tx + y$ from $\mathbb{R} \times X \times X \to X$ is continuous in each of its three variables. Here the scalar field $\mathbb{R}$ is under the usual topology. In former literature the topology $\mathcal{T}$ is called semilinear (cf. [4, 5, 7]). These assumptions are weaker than those for $X$ to be a topological vector space. The finest $\mathcal{T}$ on $X$ is formed by taking all subsets $A \subset X$ with the property that if $x_0 \in A$, $x \in X$, then there exists a $\delta > 0$ such that $tx + (1-t)x_0 \in A$ for all $t \in ]-\delta, \delta[$. In earlier literature such sets $A$ are called algebraically open [11] (cf. also [3-5, 7]).

Lemma 1. If $D$ is convex and $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex, then

(1) $f(k2^{-n}x + (1-k2^{-n})y) \leq k2^{-n}f(x) + (1-k2^{-n})f(y) + (2-2^{-n+1})\varepsilon$

for all $x, y \in D$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and $k \in \{0, 1, \ldots, 2^n\}$. 

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Proof. We proceed by induction. For $n = 1$, the inequality is clear. Assume that (1) holds for some $n \in \mathbb{N}$. Let $x, y \in D$ and $k \in \{0, 1, \ldots, 2^n+1\}$ be arbitrarily given. By appropriately labelling $x$ and $y$ we may assume that $k \leq 2^n$. Then we get
\[
f(k2^{-n-1}x + (1 - k2^{-n-1})y) = f\left(\frac{(k2^{-n}x + (1 - k2^{-n})y) + y}{2}\right)
\leq \frac{1}{2}f(k2^{-n}x + (1 - k2^{-n})y) + \frac{1}{2}f(y) + \epsilon
\leq \frac{1}{2}[k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\epsilon] + \frac{1}{2}f(y) + \epsilon
= k2^{-n-1}f(x) + (1 - k2^{-n-1})f(y) + (2 - 2^{-n})\epsilon
\]
as required. This proves the lemma.

Lemma 2. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\epsilon$-midconvex and locally bounded from above at a point $x_0 \in D$, then it is locally bounded from below at this point.

Proof. Let $U \subset D$ be an open set containing $x_0$ on which $f(x) \leq M$. Let $V := U \cap (2x_0 - U)$. Then $V$ is an open set containing $x_0$. Let $x \in V$ be given, and let $x' = 2x_0 - x$. Then $x' \in U$, and
\[
f(x_0) = f\left(\frac{x + x'}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(x') + \epsilon.
\]
Hence $f(x) \geq 2f(x_0) - f(x') - 2\epsilon \geq 2f(x_0) - M - 2\epsilon$, proving that $f$ is bounded from below on $V$.

Lemma 3. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\epsilon$-midconvex and locally bounded from above at a point of $D$, then it is locally bounded from above at every point of $D$.

Proof. Assume that $f$ is bounded from above on an open set $U \subset D$ containing $x_0$. Let $x \in D$ be arbitrarily given. Since $D$ is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x = 2^{-n}x_0 + (1 - 2^{-n})z$. Put $V := 2^{-n}U + (1 - 2^{-n})z$. Then $V$ is open and contains $x$. For every $v \in V$, $v = 2^{-n}u + (1 - 2^{-n})z$ for some $u \in U$. Hence, by Lemma 1, we get $f(v) \leq 2^{-n}f(u) + (1 - 2^{-n})f(z) + 2\epsilon$. The boundedness of $f$ from above on $V$ now follows from that of $f$ on $U$. This proves the local boundedness of $f$ from above at $x$.

Lemma 4. Let $D$ be open and convex. If $f: D \to \mathbb{R}$ is $\epsilon$-midconvex and locally bounded from below at a point of $D$, then it is locally bounded from below at every point of $D$.

Proof. Assume that $f$ is bounded from below on an open $U \subset D$ containing $x_0$, and let $x \in D$ be arbitrarily given. Since $D$ is open, there exist a point $z \in D$ and a number $n \in \mathbb{N}$ such that $x_0 = 2^{-n}x + (1 - 2^{-n})z$. Let $V := (2^nU + (1 - 2^{-n})z) \cap D$. Then $V$ is an open neighbourhood of $x$. If $v \in V$, then $u := 2^{-n}v + (1 - 2^{-n})z \in U$, and so by Lemma 1, $f(u) \leq 2^{-n}f(v) + (1 - 2^{-n})f(z) + 2\epsilon$. The boundedness of $f$ from below on $U$ now implies that of $f$ on $V$. This proves the local boundedness of $f$ from below at $x$.

Lemma 5. Let $D$ be a convex subset of $X$. If $f: D \to \mathbb{R}$ is $\epsilon$-midconvex and $\delta$-convex, then it is $2\epsilon$-convex.
Proof. Let \( x \neq y \) in \( D \) be arbitrarily fixed. By assumption
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \delta \quad \text{for all } t \in [0, 1].
\]
First, for \( t \in [0, \frac{1}{2}] \), we obtain
\[
f(tx + (1-t)y) = f(\frac{1}{2}[2tx + (1-2t)y] + \frac{1}{2}y)
\leq \frac{1}{2}f(2tx + (1-2t)y) + \frac{1}{2}f(y) + \epsilon
\leq \frac{1}{2}[2tf(x) + (1-2t)f(y) + \delta] + \frac{1}{2}f(y) + \epsilon
= tf(x) + (1-t)f(y) + \delta_1,
\]
where \( \delta_1 = \delta/2 + \epsilon \). By symmetry in \( x \) and \( y \), the above extends to all \( t \in [0, 1] \), yielding the fact that \( f \) is \( \delta_1 \)-convex. Iterating this scheme, we get that \( f \) is \( \delta_n \)-convex for \( n = 2, 3, \ldots \), where
\[
\delta_n = \frac{1}{2}\delta_{n-1} + \epsilon.
\]
Since \( \delta_n \to 2\epsilon \) as \( n \to \infty \), we obtain the conclusion that \( f \) is \( 2\epsilon \)-convex.

Theorem 1. Let \( D \subset X \) be open and convex. If \( f:D \to \mathbb{R} \) is \( \epsilon \)-midconvex and locally bounded from above at a point of \( D \), then \( f \) is \( 2\epsilon \)-convex.

Proof. By Lemmas 2 and 3, \( f \) is locally bounded from both sides at every point in \( D \). Let \( x \neq y \) be arbitrarily given in \( D \). The segment \([x, y] = \{tx + (1-t)y: t \in [0, 1]\}\) is the image of the compact interval \([0, 1]\) under the continuous map \( t \to tx + (1-t)y \), and so \([x, y]\) is compact. The local boundedness of \( f \) at every point in \( D \) implies that \( f \) is bounded on \([x, y]\), say by \( M \). This implies that the restriction of \( f \) to \([x, y]\) is \( 2M \)-convex. By Lemma 5, applied to \( f \) on \([x, y]\), we get
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\epsilon
\]
for all \( t \in [0, 1] \). As \( x, y \) are arbitrary, this proves that \( f \) is \( 2\epsilon \)-convex on \( D \).

Corollary 1. Let \( D \) be an open convex subset of \( \mathbb{R}^n \), and let \( f:D \to \mathbb{R} \) be \( \epsilon \)-midconvex. If \( f \) is bounded from above on a set \( A \subset D \) of positive Lebesgue measure, then it is \( 2\epsilon \)-convex.

Proof. Assume that \( f(x) \leq M \) for all \( x \in A \). Then
\[
f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \epsilon \leq M + \epsilon \quad \text{for all } x, y \in A.
\]
Since, by the theorem of Steinhaus, \( \frac{1}{2}(A + A) \) has nonempty interior, the local boundedness of \( f \) from above follows. Theorem 1 now yields the conclusion.

Remark. The assumption that \( D \) is open in Theorem 1 is not redundant. We give an example. Let \( D \subset \mathbb{R}^2 \) be the closed half plane \( \{(x, y) \in \mathbb{R}^2: y \geq 0\}\), and let \( f:D \to \mathbb{R} \) be given by \( f(x, y) = 0 \) if \( y > 0 \), and \( f(x, y) = |a(x)| \) if \( y = 0 \). Here \( a: \mathbb{R} \to \mathbb{R} \) is a discontinuous additive map. Then \( f \) is bounded locally at each point interior to \( D \) and is midconvex on \( D \); however, \( f \) is not convex on the \( x \)-axis and is, therefore, not convex on \( D \).

Lemma 6. Let \( I \subset \mathbb{R} \) be an interval. If \( f:I \to \mathbb{R} \) is \( \epsilon \)-midconvex on \( I \) and \( 2\epsilon \)-convex in the interior of \( I \), then \( f \) is \( 2\epsilon \)-convex on \( I \).

Proof. We may suppose that \( I \) is not degenerated, to be interesting. Let \( x \neq y \) be given in \( I \). Consider \( z = tx + (1-t)y \) for given \( t \in ]0, 1[. \) Let \( u = (x+z)/2 \)

and $v = (z + y)/2$. Then $u, v$ are interior to $I$, and $z = tu + (1-t)v$. Hence, by $2\varepsilon$-convexity in the interior of $I$, we get

$$f(z) \leq tf(u) + (1-t)f(v) + 2\varepsilon.$$ 

Since $f(u) \leq [f(x) + f(z)]/2 + \varepsilon$ and $f(v) \leq [f(z) + f(y)]/2 + \varepsilon$ by $\varepsilon$-midconvexity on $I$, we obtain

$$f(z) \leq t \left[ \frac{f(x) + f(z)}{2} + \varepsilon \right] + (1-t) \left[ \frac{f(z) + f(y)}{2} + \varepsilon \right] + 2\varepsilon.$$

This simplifies to

$$f(z) \leq tf(x) + (1-t)f(y) + 6\varepsilon.$$ 

As $t \in ]0, 1[\text{ is arbitrary, this proves that } f \text{ is } 6\varepsilon\text{-convex on } I. \text{ By Lemma 5, } f \text{ is } 2\varepsilon\text{-convex on } I.$

**Theorem 2.** Let $D \subset X$ be convex, and suppose that the boundary of $D$ contains no proper segment $[a, b] = \{ta + (1-t)b: t \in [0, 1]\}$ where $a \neq b$ in $D$. If $f: D \to \mathbb{R}$ is $\varepsilon$-midconvex and is locally bounded from above at a point interior to $D$, then $f$ is $2\varepsilon$-convex.

**Proof.** By Lemma 3, applied to the restriction of $f$ to the interior of $D$, we get the local boundedness of $f$ from above at every interior point of $D$. To show that $f$ is $2\varepsilon$-convex, we need to show that for an arbitrary given proper segment $[a, b] \subset D$, $f$ is $2\varepsilon$-convex on $[a, b]$. Consider pulling $[a, b]$ back to $[0, 1]$ via $g: [0, 1] \to [a, b]$, $g(t) = ta + (1-t)b$. Also consider $\bar{f} := f \circ g$. Since $[a, b]$ contains interior points of $D$, $\bar{f}$ is locally bounded from above at some interior point of $[0, 1]$. Applying Theorem 1 to $\bar{f}$, we get that $\bar{f}$ is $2\varepsilon$-convex in $[0, 1]$. Applying Lemma 6, we get the $2\varepsilon$-convexity of $\bar{f}$ on $[0, 1]$. This in turn yields that $f$ is $2\varepsilon$-convex on $[a, b]$.

**Example.** Let $D$ be a closed ball in $\mathbb{R}^n$ (with the usual topology). Then every $\varepsilon$-midconvex function $f: D \to \mathbb{R}$, locally bounded from above at a point interior to $D$, must be $2\varepsilon$-convex. This observation extends to closed balls in a strictly convex real normed linear space.

**Lemma 7.** Let $D \subset X$ be open and convex. If $f: D \to \mathbb{R}$ is $\varepsilon$-Wright-convex and locally bounded from below at a point of $D$, then it is $2\varepsilon$-convex.

**Proof.** Let $x \neq y$ in $D$ be arbitrarily fixed. We need to show that $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\varepsilon$ for all $t \in [0, 1]$. This is an observation on the one-dimensional line passing $x$ and $y$; we can formally pull the problem back to the real field as follows.

Consider $E = \{ t \in \mathbb{R}: tx + (1-t)y \in D \}$ and $g: E \to \mathbb{R}$ given by $g(t) = f(tx + (1-t)y)$. Since $D$ is open and convex, so is $E \subset \mathbb{R}$. By Lemma 4, the local boundedness of $f$ from below at one point extends to every point of $D$, leading to the local boundedness of $g$ from below at every point of $E$. Since $[0, 1]$ is compact, $g$ is bounded from below on $[0, 1]$. The $\varepsilon$-Wright-convexity passes onto $g$. In particular, we have

$$g(1-t) + g(t) \leq g(0) + g(1) + 2\varepsilon \text{ for all } t \in [0, 1].$$

As $g(t)$ is bounded from below over all $t \in [0, 1]$, the above implies that $g(1-t)$ is bounded from above over all $t \in [0, 1]$. Thus $g$ is bounded from
above on \([0, 1]\). It follows from Theorem 1 that \(g\) is \(2\varepsilon\)-convex on \(E\). Hence
\[
f(tx + (1-t)y) = g(t) = g(t \cdot 1 + (1-t) \cdot 0) \\
\leq tg(1) + (1-t)g(0) + 2\varepsilon = tf(x) + (1-t)f(y) + 2\varepsilon,
\]
as required. This proves the lemma.

**Theorem 3.** Let \(D \subset X\) be convex. If \(f: D \to \mathbb{R}\) is \(\varepsilon\)-Wright-convex and locally bounded from below at an interior point of \(D\), then it is \(2\varepsilon\)-convex.

**Proof.** Suppose \(x_0\) is an interior point of \(D\) and \(f\) is locally bounded from below at \(x_0\). From Lemma 7, it follows that \(f\) is \(2\varepsilon\)-convex in the interior of \(D\). Let \([x, y]\) with \(x \neq y\) be a given proper segment of \(D\). We need to show that \(f\) is \(2\varepsilon\)-convex on \([x, y]\). There are two possibilities. First consider the case where \([x, y]\) contains an interior point of \(D\). Then, by Lemma 4, \(f\) is locally bounded from below at a point of \([x, y]\). Evidently, this implies it is locally bounded from below at a point in \([x, y]\) := \(\{tx + (1-t)y : 0 < t < 1\}\). Applying Lemma 7 to \(f\) on \([x, y]\), or to its pull back on \([0, 1]\) if necessary, we obtain that \(f\) is \(2\varepsilon\)-convex on \([x, y]\). Further, by Lemma 6, we obtain that \(f\) is \(2\varepsilon\)-convex on \([x, y]\).

Second, consider the case where \([x, y]\) is on the boundary of \(D\). In this case consider the triangle with vertices \(x_0, x,\) and \(y\). By \(\varepsilon\)-midconvexity of \(f\) we get
\[
f\left(\frac{x_0 + z}{2}\right) \leq \frac{1}{2}f(x_0) + \frac{1}{2}f(z) + \varepsilon \quad \text{for all } z \in [x, y];
\]
but the segment \(\{\frac{1}{2}x_0 + \frac{1}{2}z : z \in [x, y]\}\) is in the interior of \(D\) and is compact; thus \(f\) is bounded from below on this segment. The above inequality implies that \(f\) is bounded from below on \([x, y]\). Thus, applying Lemma 7, we first get that \(f\) is \(2\varepsilon\)-convex on \([x, y]\) and, further by Lemma 6, obtain that \(f\) is \(2\varepsilon\)-convex on \([x, y]\). This completes the proof.

**Remarks.** The above results remain valid when openness of a convex set \(D\) is replaced by its openness relative to the manifold it generates. Theorems 1 and 3 are most forceful when the topology is the topology of algebraically open sets. Theorem 1 reduces to a result obtained by Kominek [3, Theorem 2] when \(\varepsilon = 0\). The ratio \(2\varepsilon\) in these two theorems is the best possible, as the following example illustrates. Let \(f: \mathbb{R} \to \mathbb{R}, f(x) = 0\) for \(x \leq 0\), and \(f(x) = 1\) for \(x > 0\). Then \(f\) is \(\varepsilon\)-midconvex with lowest \(\varepsilon = 1/2\). It is \(\varepsilon\)-convex with lowest \(\varepsilon = 1\). In Theorem 1 boundedness from above cannot be replaced by boundedness from below. For example, \(f(x) = |a(x)|\), where \(a: \mathbb{R} \to \mathbb{R}\) is a discontinuous additive map, is midconvex on \(\mathbb{R}\), and is locally bounded from below. Yet \(f\) is not convex.

**References**


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