ON APPROXIMATELY CONVEX FUNCTIONS

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Abstract. The Bernstein-Doetsch theorem on midconvex functions is extended to approximately midconvex functions and to approximately Wright convex functions.

Let $X$ be a real vector space, $D$ be a convex subset of $X$, and $\varepsilon$ be a nonnegative constant. A function $f : D \to \mathbb{R}$ is said to be

$\varepsilon$-convex if $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$ (cf. [2]);

$\varepsilon$-Wright-convex if $f(tx + (1 - t)y) + f((1 - t)x + ty) \leq f(x) + f(y) + 2\varepsilon$ for all $x, y \in D$ and $t \in [0, 1]$;

$\varepsilon$-midconvex if $f(\frac{tx + ty}{2}) \leq \frac{1}{2}f(x) + f(y)) + \varepsilon$ for all $x, y \in D$.

Notice that $\varepsilon$-convexity implies $\varepsilon$-Wright-convexity, which in turn implies $\varepsilon$-midconvexity, but not the converse. The usual notions of convexity, Wright-convexity, and midconvexity correspond to the case $\varepsilon = 0$. A comprehensive review on this subject can be found in [1, 6, 8–10]. The Bernstein-Doetsch theorem relates local boundedness, midconvexity, and convexity (cf. [6, 10]). In order to extend this result to approximately midconvex functions, we first specify the assumptions on the topology $\mathcal{T}$ to be imposed on $X$: the map $(t, x, y) \mapsto tx + y$ from $\mathbb{R} \times X \times X \to X$ is continuous in each of its three variables. Here the scalar field $\mathbb{R}$ is under the usual topology. In former literature the topology $\mathcal{T}$ is called semilinear (cf. [4, 5, 7]). These assumptions are weaker than those for $X$ to be a topological vector space. The finest $\mathcal{T}$ on $X$ is formed by taking all subsets $A \subseteq X$ with the property that if $x_0 \in A$, $x \in X$, then there exists a $\delta > 0$ such that $tx + (1 - t)x_0 \in A$ for all $t \in [-\delta, \delta]$. In earlier literature such sets $A$ are called algebraically open [11] (cf. also [3–5, 7]).

Lemma 1. If $D$ is convex and $f : D \to \mathbb{R}$ is $\varepsilon$-midconvex, then

$$f(k2^{-n}x + (1 - k2^{-n})y) \leq k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\varepsilon$$

for all $x, y \in D$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, and $k \in \{0, 1, \ldots, 2^n\}$. 

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Proof. We proceed by induction. For \( n = 1 \), the inequality is clear. Assume that (1) holds for some \( n \in \mathbb{N} \). Let \( x, y \in D \) and \( k \in \{0, 1, \ldots, 2^{n+1}\} \) be arbitrarily given. By appropriately labelling \( x \) and \( y \) we may assume that \( k \leq 2^n \). Then we get

\[
f(k2^{-n-1}x + (1 - k2^{-n-1})y) = f\left(\frac{(k2^{-n}x + (1 - k2^{-n})y) + y}{2}\right)
\]

\[
\leq \frac{1}{2} f(k2^{-n}x + (1 - k2^{-n})y) + \frac{1}{2} f(y) + \varepsilon
\]

\[
\leq \frac{1}{2} [k2^{-n}f(x) + (1 - k2^{-n})f(y) + (2 - 2^{-n+1})\varepsilon] + \frac{1}{2} f(y) + \varepsilon
\]

\[
= k2^{-n-1}f(x) + (1 - k2^{-n-1})f(y) + (2 - 2^{-n})\varepsilon
\]

as required. This proves the lemma.

Lemma 2. Let \( D \) be open and convex. If \( f: D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and locally bounded from above at a point \( x_0 \in D \), then it is locally bounded from below at this point.

Proof. Let \( U \subset D \) be an open set containing \( x_0 \) on which \( f(x) \leq M \). Let \( V := U \cap (2x_0 - U) \). Then \( V \) is an open set containing \( x_0 \). Let \( x \in V \) be given, and let \( x' = 2x_0 - x \). Then \( x' \in U \), and

\[
f(x_0) = f\left(\frac{x + x'}{2}\right) \leq \frac{1}{2} f(x) + \frac{1}{2} f(x') + \varepsilon.
\]

Hence \( f(x) \geq 2f(x_0) - f(x') - 2\varepsilon \geq 2f(x_0) - M - 2\varepsilon \), proving that \( f \) is bounded from below on \( V \).

Lemma 3. Let \( D \) be open and convex. If \( f: D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and locally bounded from above at a point of \( D \), then it is locally bounded from above at every point of \( D \).

Proof. Assume that \( f \) is bounded from above on an open set \( U \subset D \) containing \( x_0 \). Let \( x \in D \) be arbitrarily given. Since \( D \) is open, there exist a point \( z \in D \) and a number \( n \in \mathbb{N} \) such that \( x = 2^{-n}x_0 + (1 - 2^{-n})z \). Put \( V := 2^{-n}U + (1 - 2^{-n})z \). Then \( V \) is open and contains \( x \). For every \( v \in V \), \( v = 2^{-n}u + (1 - 2^{-n})z \) for some \( u \in U \). Hence, by Lemma 1, we get \( f(v) \leq 2^{-n}f(u) + (1 - 2^{-n})f(z) + 2\varepsilon \). The boundedness of \( f \) from above on \( V \) now follows from that of \( f \) on \( U \). This proves the local boundedness of \( f \) from above at \( x \).

Lemma 4. Let \( D \) be open and convex. If \( f: D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and locally bounded from below at a point of \( D \), then it is locally bounded from below at every point of \( D \).

Proof. Assume that \( f \) is bounded from below on an open \( U \subset D \) containing \( x_0 \), and let \( x \in D \) be arbitrarily given. Since \( D \) is open, there exist a point \( z \in D \) and a number \( n \in \mathbb{N} \) such that \( x_0 = 2^{-n}x_0 + (1 - 2^{-n})z \). Let \( V := (2^nU + (1 - 2^n)z) \cap D \). Then \( V \) is an open neighbourhood of \( x \). If \( v \in V \), then \( u := 2^{-n}v + (1 - 2^{-n})z \in U \), and so by Lemma 1, \( f(u) \leq 2^{-n}f(v) + (1 - 2^{-n})f(z) + 2\varepsilon \). The boundedness of \( f \) from below on \( U \) now implies that of \( f \) on \( V \). This proves the local boundedness of \( f \) from below at \( x \).

Lemma 5. Let \( D \) be a convex subset of \( X \). If \( f: D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and \( \delta \)-convex, then it is \( 2\varepsilon \)-convex.
Proof. Let \( x \neq y \) in \( D \) be arbitrarily fixed. By assumption
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \delta \quad \text{for all } t \in [0, 1].
\]
First, for \( t \in [0, \frac{1}{2}] \), we obtain
\[
f(tx + (1 - t)y) = f(\frac{1}{2}[2tx + (1 - 2t)y] + \frac{1}{2}y)
\leq \frac{1}{2}f(2tx + (1 - 2t)y) + \frac{1}{2}f(y) + \epsilon
\leq \frac{1}{2}[2tf(x) + (1 - 2t)f(y) + \delta] + \frac{1}{2}f(y) + \epsilon
= tf(x) + (1 - t)f(y) + \delta_1,
\]
where \( \delta_1 = \frac{\delta}{2} + \epsilon \). By symmetry in \( x \) and \( y \), the above extends to all \( t \in [0, 1] \), yielding the fact that \( f \) is \( \delta_1 \)-convex. Iterating this scheme, we get that \( f \) is \( \delta_n \)-convex for \( n = 2, 3, \ldots \), where
\[
\delta_n = \frac{1}{2}\delta_{n-1} + \epsilon.
\]
Since \( \delta_n \to 2\epsilon \) as \( n \to \infty \), we obtain the conclusion that \( f \) is \( 2\epsilon \)-convex.

Theorem 1. Let \( D \subset X \) be open and convex. If \( f:D \to \mathbb{R} \) is \( \epsilon \)-midconvex and locally bounded from above at a point of \( D \), then \( f \) is \( 2\epsilon \)-convex.

Proof. By Lemmas 2 and 3, \( f \) is locally bounded from both sides at every point in \( D \). Let \( x \neq y \) be arbitrarily given in \( D \). The segment \([x, y] = \{tx + (1-t)y : t \in [0, 1]\}\) is the image of the compact interval \([0, 1]\) under the continuous map \( t \to tx + (1-t)y \), and so \([x, y]\) is compact. The local boundedness of \( f \) at every point in \( D \) implies that \( f \) is bounded on \([x, y]\), say by \( M \). This implies that the restriction of \( f \) to \([x, y]\) is \( 2M \)-convex. By Lemma 5, applied to \( f \) on \([x, y]\), we get
\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + 2\epsilon \quad \text{for all } t \in [0, 1].
\]
As \( x, y \) are arbitrary, this proves that \( f \) is \( 2\epsilon \)-convex on \( D \).

Corollary 1. Let \( D \) be an open convex subset of \( \mathbb{R}^n \), and let \( f:D \to \mathbb{R} \) be \( \epsilon \)-midconvex. If \( f \) is bounded from above on a set \( A \subset D \) of positive Lebesgue measure, then it is \( 2\epsilon \)-convex.

Proof. Assume that \( f(x) \leq M \) for all \( x \in A \). Then
\[
f \left( \frac{x+y}{2} \right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) + \epsilon \leq M + \epsilon \quad \text{for all } x, y \in A.
\]
Since, by the theorem of Steinhaus, \( \frac{1}{2}(A + A) \) has nonempty interior, the local boundedness of \( f \) from above follows. Theorem 1 now yields the conclusion.

Remark. The assumption that \( D \) is open in Theorem 1 is not redundant. We give an example. Let \( D \subset \mathbb{R}^2 \) be the closed half plane \( \{(x, y) \in \mathbb{R}^2 : y \geq 0\}\), and let \( f:D \to \mathbb{R} \) be given by \( f(x, y) = 0 \) if \( y > 0 \), and \( f(x, y) = |a(x)| \) if \( y = 0 \). Here \( a: \mathbb{R} \to \mathbb{R} \) is a discontinuous additive map. Then \( f \) is bounded locally at each point interior to \( D \) and is midconvex on \( D \); however, \( f \) is not convex on the \( x \)-axis and is, therefore, not convex on \( D \).

Lemma 6. Let \( I \subset \mathbb{R} \) be an interval. If \( f:I \to \mathbb{R} \) is \( \epsilon \)-midconvex on \( I \) and \( 2\epsilon \)-convex in the interior of \( I \), then \( f \) is \( 2\epsilon \)-convex on \( I \).

Proof. We may suppose that \( I \) is not degenerated, to be interesting. Let \( x \neq y \) be given in \( I \). Consider \( z = tx + (1-t)y \) for given \( t \in [0, 1[. \) Let \( u = (x+z)/2 \)
and \( v = (z+y)/2 \). Then \( u, v \) are interior to \( I \), and \( z = tu + (1-t)v \). Hence, by \( 2\varepsilon \)-convexity in the interior of \( I \), we get

\[
f(z) \leq tf(u) + (1-t)f(v) + 2\varepsilon.
\]

Since \( f(u) \leq \frac{f(x) + f(z)}{2} + \varepsilon \) and \( f(v) \leq \frac{f(z) + f(y)}{2} + \varepsilon \) by \( \varepsilon \)-midconvexity on \( I \), we obtain

\[
f(z) \leq t \left[ \frac{f(x) + f(z)}{2} + \varepsilon \right] + (1-t) \left[ \frac{f(z) + f(y)}{2} + \varepsilon \right] + 2\varepsilon.
\]

This simplifies to

\[
f(z) \leq tf(x) + (1-t)f(y) + 6\varepsilon.
\]

As \( t \in [0, 1] \) is arbitrary, this proves that \( f \) is \( 6\varepsilon \)-convex on \( I \). By Lemma 5, \( f \) is \( 2\varepsilon \)-convex on \( I \).

**Theorem 2.** Let \( D \subset X \) be convex, and suppose that the boundary of \( D \) contains no proper segment \([a, b] = \{ta + (1-t)b : t \in [0, 1]\}\) where \( a \neq b \) in \( D \). If \( f:D \to \mathbb{R} \) is \( \varepsilon \)-midconvex and is locally bounded from above at a point interior to \( D \), then \( f \) is \( 2\varepsilon \)-convex.

**Proof.** By Lemma 3, applied to the restriction of \( f \) to the interior of \( D \), we get the local boundedness of \( f \) from above at every interior point of \( D \). To show that \( f \) is \( 2\varepsilon \)-convex, we need to show that for an arbitrary given proper segment \([a, b] \subset D \), \( f \) is \( 2\varepsilon \)-convex on \([a, b]\). Consider pulling \([a, b]\) back to \([0, 1]\) via \( g: [0, 1] \to [a, b] \), \( g(t) = ta + (1-t)b \). Also consider \( \tilde{f} = f \circ g \). Since \([a, b]\) contains interior points of \( D \), \( \tilde{f} \) is locally bounded from above at some interior point of \([0, 1]\). Applying Theorem 1 to \( \tilde{f} \), we get that \( \tilde{f} \) is \( 2\varepsilon \)-convex in \([0, 1]\). Applying Lemma 6, we get the \( 2\varepsilon \)-convexity of \( \tilde{f} \) on \([0, 1]\). This in turn yields that \( f \) is \( 2\varepsilon \)-convex on \([a, b]\).

**Example.** Let \( D \) be a closed ball in \( \mathbb{R}^n \) (with the usual topology). Then every \( \varepsilon \)-midconvex function \( f:D \to \mathbb{R} \), locally bounded from above at a point interior to \( D \), must be \( 2\varepsilon \)-convex. This observation extends to closed balls in a strictly convex real normed linear space.

**Lemma 7.** Let \( D \subset X \) be open and convex. If \( f:D \to \mathbb{R} \) is \( \varepsilon \)-Wright-convex and locally bounded from below at a point of \( D \), then it is \( 2\varepsilon \)-convex.

**Proof.** Let \( x \neq y \) in \( D \) be arbitrarily fixed. We need to show that \( f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + 2\varepsilon \) for all \( t \in [0, 1] \). This is an observation on the one-dimensional line passing \( x \) and \( y \); we can formally pull the problem back to the real field as follows.

Consider \( E = \{t \in \mathbb{R} : tx + (1-t)y \in D\} \) and \( g:E \to \mathbb{R} \) given by \( g(t) = f(tx + (1-t)y) \). Since \( D \) is open and convex, so is \( E \subset \mathbb{R} \). By Lemma 4, the local boundedness of \( f \) from below at one point extends to every point of \( D \), leading to the local boundedness of \( g \) from below at every point of \( E \). Since \([0, 1]\) is compact, \( g \) is bounded from below on \([0, 1]\). The \( \varepsilon \)-Wright-convexity passes onto \( g \). In particular, we have

\[
g(1-t) + g(t) \leq g(0) + g(1) + 2\varepsilon \quad \text{for all } t \in [0, 1].
\]

As \( g(t) \) is bounded from below over all \( t \in [0, 1] \), the above implies that \( g(1-t) \) is bounded from above over all \( t \in [0, 1] \). Thus \( g \) is bounded from
above on $[0, 1]$. It follows from Theorem 1 that $g$ is $2\varepsilon$-convex on $E$. Hence
\[
f(tx + (1-t)y) = g(t) = g(t \cdot 1 + (1-t) \cdot 0) \\
\leq tg(1) + (1-t)g(0) + 2\varepsilon = tf(x) + (1-t)f(y) + 2\varepsilon,
\]
as required. This proves the lemma.

**Theorem 3.** Let $D \subset X$ be convex. If $f:D \to \mathbb{R}$ is $\varepsilon$-Wright-convex and locally bounded from below at an interior point of $D$, then it is $2\varepsilon$-convex.

**Proof.** Suppose $x_0$ is an interior point of $D$ and $f$ is locally bounded from below at $x_0$. From Lemma 7, it follows that $f$ is $2\varepsilon$-convex in the interior of $D$. Let $[x, y]$ with $x \neq y$ be a given proper segment of $D$. We need to show that $f$ is $2\varepsilon$-convex on $[x, y]$. There are two possibilities. First consider the case where $[x, y]$ contains an interior point of $D$. Then, by Lemma 4, $f$ is locally bounded from below at a point of $[x, y]$. Evidently, this implies it is locally bounded from below at a point in $]x, y[ := \{tx + (1-t)y: 0 < t < 1\}$. Applying Lemma 7 to $f$ on $]x, y[$ or to its pull back on $]0, 1[$ if necessary, we obtain that $f$ is $2\varepsilon$-convex on $]x, y[$. Further, by Lemma 6, we obtain that $f$ is $2\varepsilon$-convex on $[x, y]$.

Second, consider the case where $[x, y]$ is on the boundary of $D$. In this case consider the triangle with vertices $x_0, x, y$. By $\varepsilon$-midconvexity of $f$ we get
\[
f\left(\frac{x_0 + z}{2}\right) \leq \frac{1}{2}f(x_0) + \frac{1}{2}f(z) + \varepsilon \quad \text{for all } z \in [x, y];
\]
but the segment $\{\frac{1}{2}x_0 + \frac{1}{2}z: z \in [x, y]\}$ is in the interior of $D$ and is compact; thus $f$ is bounded from below on this segment. The above inequality implies that $f$ is bounded from below on $[x, y]$. Thus, applying Lemma 7, we first get that $f$ is $2\varepsilon$-convex on $]x, y[$ and, further by Lemma 6, obtain that $f$ is $2\varepsilon$-convex on $[x, y]$. This completes the proof.

**Remarks.** The above results remain valid when openness of a convex set $D$ is replaced by its openness relative to the manifold it generates. Theorems 1 and 3 are most forceful when the topology is the topology of algebraically open sets. Theorem 1 reduces to a result obtained by Kominek [3, Theorem 2] when $\varepsilon = 0$. The ratio $2\varepsilon$ in these two theorems is the best possible, as the following example illustrates. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ for $x \leq 0$, and $f(x) = 1$ for $x > 0$. Then $f$ is $\varepsilon$-midconvex with lowest $\varepsilon = 1/2$. It is $\varepsilon$-convex with lowest $\varepsilon = 1$. In Theorem 1 boundedness from above cannot be replaced by boundedness from below. For example, $f(x) = |a(x)|$, where $a: \mathbb{R} \to \mathbb{R}$ is a discontinuous additive map, is midconvex on $\mathbb{R}$, and is locally bounded from below. Yet $f$ is not convex.

**References**


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