ON MAPPINGS WITH INTEGRABLE DILATATION

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Abstract. A factorization of Stoilow's type has been obtained for mappings in $\mathbb{R}^2$ with integrable dilatation.

0. Introduction

For $\Omega$ a domain in $\mathbb{R}^n$ (an open and connected set), we consider a mapping $f: \Omega \to \mathbb{R}^n$ of the Sobolev class $W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ with nonnegative Jacobian, $J(x, f) \geq 0$ a.e. We say that $f$ has finite dilatation if

\begin{equation}
|Df(x)|^n \leq K(x)J(x, f) \quad \text{a.e.}
\end{equation}

where $1 \leq K(x) < \infty$ for almost every $x \in \Omega$ and $|Df(x)|$ denotes the norm of the differential $Df(x): \mathbb{R}^n \to \mathbb{R}^n$.

In recent developments of the nonlinear elasticity theory [Ba, Š], there have been intensive studies of the analytic and geometric properties of such mappings. It is known that the condition $f \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ does not guarantee that $f$ is continuous, but it does if $f$ has finite dilatation [VG], see also [Ma] for a simpler proof. To state our result we need some definitions.

The dilatation quotient at the points $x \in \Omega$ with $J(x, f) \neq 0$ is defined by

\begin{equation}
K(x, f) = \frac{|Df(x)|^n}{J(x, f)} \geq 1.
\end{equation}

If $J(x, f) = 0$, then $Df(x) = 0$, and in this case we put $K(x, f) = 1$ a.e. Therefore the dilatation function $K(\cdot, f): \Omega \to [1, \infty)$ is defined almost everywhere in $\Omega$. A mapping $f \in W_{\text{loc}}^{1,n}(\Omega, \mathbb{R}^n)$ is said to be $K$-quasi-regular, $1 \leq K < \infty$, if $K(x, f) \leq K$ a.e. If, in addition, $f$ is a homeomorphism, we say that $f$ is $K$-quasi-conformal.

A well-known result in the theory of quasi-regular mappings [Re] states that if $K(\cdot, f) \in L^\infty(\Omega)$, then $f$ is either constant or an open mapping. In two

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dimensions this fact has already been recognized by Bojarski [Bo1, Bo2], who has proved Stoilow's type factorization

\[ f = \varphi \circ h^{-1}, \]

with \( h : \Omega' \to \Omega \) a homeomorphism (quasi-conformal mapping) and \( \varphi : \Omega \to \mathbb{R}^2 \) a holomorphic function.

In this note we prove that this factorization remains valid for 2-dimensional mappings whose dilatation function is only assumed to be integrable. Such mappings are, therefore, open and discrete.

**Theorem 1.** Let \( \Omega \) be a bounded domain in the complex plane \((\mathbb{C}, d\sigma(z))\) and, \( f \in W^{1,2}(\Omega, \mathbb{C}) \) with \( J(z, f) \geq 0 \) and \( K(\cdot, f) \in L^1(\Omega) \). Then there exists a homeomorphism \( h \in W^{1,2}(\Omega', \Omega) \) and a holomorphic function \( \varphi \in W^{1,2}(\Omega', \mathbb{C}) \) such that

\[ f = \varphi \circ h^{-1}. \]

Moreover,

\[ \int_{\Omega'} |Dh(\omega)|^2 \, d\sigma(\omega) \leq \int_{\Omega} K(z, f) \, d\sigma(z) \]

and

\[ \int_{\Omega'} |\varphi'(\omega)|^2 \, d\sigma(\omega) \leq \int_{\Omega} \left| \frac{\partial}{\partial z} f(z) \right|^2 \, d\sigma(z). \]

One can ask whether some integrability condition on the dilatation function of a mapping \( f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n) \) with positive Jacobian implies openness also in dimension \( n > 2 \). The arguments we have used in the proof of Theorem 1 suggest the following.

**Conjecture 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and \( f \in W^{1,n}(\Omega, \mathbb{R}^n) \) with \( J(x, f) \geq 0 \) and \( K(\cdot, f) \in L^{n-1}(\Omega) \). Then \( f \) is either constant or an open mapping.

This has already been shown under additional assumptions about the boundary values of \( f \) [Ba, Š]. The general case still remains open.¹

1. **Preliminaries**

We need a few of the fundamental properties of the quasi-regular mapping in \( \mathbb{R}^n \). Let us recall the chain rule for differentiation of the composite functions [BI].

**Lemma 1.1.** Let \( f \in W^{1,n}_{\text{loc}}(\Omega, \Omega') \) be a quasi-regular mapping and let \( \varphi \in W^{1,n}_{\text{loc}}(\Omega') \). Then \( \varphi \circ f \in W^{1,n}_{\text{loc}}(\Omega) \), and

\[ D(\varphi \circ f)(x) = (D\varphi)(f(x)) \circ Df(x) \]

for almost every \( x \in \Omega \).

The next result concerns the change of variables in a multiple integral [BI, Re, Ri].

¹Very recently J. Heinonen and P. Koskela confirmed this conjecture for \( f \) a “quasi-light mapping” with \( K(\cdot, f) \in L^p(\Omega) \) and \( p > n - 1 \), and most recently for \( f \in W^{1,p}_{\text{loc}}(\Omega, \mathbb{R}^n) \) with \( p > n \) and \( K \in L^{p/(n-1)(p-n)}_{\text{loc}}(\Omega) \).
Lemma 1.2. Let $f: \Omega \to \Omega'$ be quasi-conformal and $u \in L^1(\Omega')$. Then $u(f(x)) J(x, f) \in L^1(\Omega)$, and we have

$$\int_{\Omega'} u(y) \, dy = \int_{\Omega} u(f(x)) J(x, f) \, dx.$$  

With the aid of these two lemmas we easily arrive at an estimate of the $L^n$-norm of the differential of the inverse mapping $h = f^{-1}: \Omega' \to \Omega$ in terms of the dilatation function of $f$.

Lemma 1.3. Let $f: \Omega \to \Omega'$ be a quasi-conformal mapping of bounded domains $\Omega, \Omega' \subset \mathbb{R}^n$, and let $h: \Omega' \to \Omega$ denote the inverse mapping. Then

$$\int_{\Omega'} |Dh(y)|^n \, dy \leq \int_{\Omega} K^{-1}(x, f) \, dx.$$  

Proof. We have

$$\int_{\Omega'} |Dh|^n \, dy = \int_{\Omega} |D(f)^{-1}(x)|^n J(x, f) \, dx = \int_{\Omega} |adj \; Df(x)|^n J^{1-n}(x, f) \, dx$$

$$\leq \int_{\Omega} |Df(x)|^{n(n-1)} J^{1-n}(x, f) \, dx = \int_{\Omega} K^{n-1}(x, f) \, dx,$$

as desired.

This is why we assumed in Conjecture 1 that $K(\cdot, f) \in L^{n-1}(\Omega)$. The last prerequisite deals with the concept and properties of monotone mappings.

We refer to the article of McAuley [McA], in which this subject is well covered by a series of papers.

Let $X$ and $Y$ be compact metric spaces. A continuous mapping $h$ from $X$ onto $Y$ is said to be monotone if for each $y \in Y$ the set $f^{-1}(y)$ is connected. Actually, as shown by Whyburn, this implies that $f^{-1}(C)$ is connected for each connected set $C$ in $Y$.

We shall use the following result of Kuratowski, Lacher, and Whyburn [McA].

Lemma 1.4. If $Y$ is locally connected, then the set of all monotone mappings from $X$ onto $Y$ is closed in $C(X, Y)$. The latter stands for the space of all continuous mappings of $X$ into $Y$ with the topology of uniform convergence.

In our application of this result $X$ and $Y$ will be the 2-spheres, in which case Lemma 1.4 is an elementary exercise.

2. The Beltrami equation

The space $\mathbb{R}^2$ will be identified with the complex plane $\mathbb{C}$, where the area element is denoted by $d\sigma(z) = dx \, dy$, $z = x + iy$. For $a \in \mathbb{C}$ and $r > 0$, we define the open disk $B(a, r) = \{z : |z - a| < r\}$ and its boundary $S(a, r) = \{z : |z - a| = r\}$.

On the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ we introduce the chordal distance

$$d(a, \infty) = \frac{2}{\sqrt{1 + |a|^2}}$$

and

$$d(a, b) = \frac{2|a - b|}{\sqrt{1 + |a|^2} \sqrt{1 + |b|^2}}$$

if $a, b \neq \infty$. 

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Thus $\hat{C}$ is a compact metric space. Recall that the chordal distance $d$ is inherited from the Euclidean metric on the 2-sphere via the stereographic projection. It is clear that $d$ restricted to $C$ induces the same topology as does the Euclidean metric.

We shall make use of the Cauchy-Riemann derivatives. For $f \in W^{1, p}_{\text{loc}}(\Omega, \mathbb{C})$, where $\Omega$ is an open subset of $C$, the derivatives are defined by

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right).$$

It is straightforward to check the formulas

$$J(z, f) = |f_z|^2 - |f_{\bar{z}}|^2 \quad \text{and} \quad |Df(z)| = |f_z| + |f_{\bar{z}}|.$$

Our proof of Theorem 1 will rest on the existence theorem for the Beltrami equation.

**Proposition 2.1** (Bojarski [Bo1, Bo2]). Let $\mu$ be an arbitrary measurable function with compact support and $\|\mu\|_{\infty} < 1$. Then, for some $p > 2$, there exists a unique solution $f \in W^{1, p}_{\text{loc}}(\mathbb{C}, \mathbb{C})$ of the Beltrami equation

$$f_z(z) = \mu(z)f_{\bar{z}}(z) \quad \text{such that} \quad f(0) = 0 \quad \text{and} \quad 1 - f_z \in L^p(\mathbb{C}).$$

This is what we call the normal solution of (2.1) The coefficient $\mu$ is referred to as the complex dilatation of $f$. The normal solution is a quasi-conformal homeomorphism of the extended complex plane, analytic outside the support of $\mu$, and its Taylor expansion at infinity takes the form $f(z) = z + a_1 z^{-1} + a_2 z^{-2} = \cdots$. See also [A, L, LV].

For results concerning the existence of solutions of (2.1) with $\|\mu\|_{\infty} = 1$, we refer to David [D]; see also [P].

From now on we confine ourselves to those $\mu$ which are supported in the unit disk $B = \{z; |z| < 1\}$. The purpose of this section is to establish uniform estimates for the inverse mapping $h = f^{-1} : \hat{C} \to \hat{C}$. First, recall the inequality

$$\int_{E} |Dh(\omega)|^2 d\sigma(\omega) \leq \int_{h(E)} K(z, f) d\sigma(z)$$

for each measurable set $E \subset \mathbb{C}$. Here, the dilatation function of $f$ can be expressed in terms of $\mu$ as

$$K(z, f) = \frac{|Df(z)|^2}{J(z, f)} = \frac{1}{1 - |\mu(z)|^2}.$$

**Proposition 2.2.** Let $B_r = B(0, r)$, $r > 1$, and $\xi$, $\zeta \in B_r$ be such that $|\xi - \zeta| < 2$. Then

$$|h(\xi) - h(\zeta)|^2 \log \frac{4}{|\xi - \zeta|^2} \leq \pi \int_{B_{r, 3}} K(z, f) d\sigma(z).$$

The chordal distance from $h(\xi)$ to $\infty$ is estimated independently of $\mu$ as

$$d(h(\xi), \infty) \leq \frac{10}{1 + |\xi|}$$

for all $\xi \in \hat{C}$. 

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Proof. Take notice that $f$ maps $\widehat{\mathbb{C}} - \mathbb{B}$ univalently into $\widehat{\mathbb{C}} - \{0\}$. In view of the Koebe distortion inequality, we can write $|z| + |z|^{-1} - 2 \leq |f(z)| \leq |z| + |z|^{-1} + 2$, for all $z$ with $|z| > 1$; see, for example, [M]. Hence, $B_{r+1} \subset f(B_{r+3})$, which is equivalent to

$$h(B_{r+1}) \subset B_{r+3}.$$  

We also infer that

$$|f(z)| \leq 4|z| \quad \text{for } |z| \geq 1.$$  

To prove inequality (2.3) we set $|\xi - \zeta| = 2\delta < 2$ and $a = \frac{1}{2}(\xi + \zeta) \in B_r$. Obviously, $S(a, \delta) \subset B(a, t) \subset B_{r+1}$ if $\delta < t < 1$. Since $h$ is a homeomorphism, by the maximum principle, one can find points $\xi', \zeta' \in S(a, t)$ such that $|h(\xi) - h(\zeta)| \leq |h(\xi') - h(\zeta')|$. The latter is easily estimated by the integral of $|Dh|$ over the circle $S(a, t) :$

$$|h(\xi) - h(\zeta)| \leq \frac{1}{2} \int_{S(a, t)} |Dh(\omega)||d\omega|$$

for almost every $t \in (\delta, 1)$. By Hölder's inequality we obtain

$$t^{-1}|h(\xi) - h(\zeta)|^2 \leq \frac{\pi}{2} \int_{S(a, t)} |Dh(\omega)|^2|d\omega|.$$ 

Integrating with respect to $t \in (\delta, 1)$, by Fubini's theorem, we find that

$$-2|h(\xi) - h(\zeta)|^2 \log \delta \leq \pi \int_{B(a, 1)} |Dh(\omega)|^2d\sigma(\omega)$$

$$\leq \pi \int_{B_{r+1}} |Dh(\omega)|^2d\sigma(\omega).$$

This, together with (2.2) and (2.5), yields

$$|h(\xi) - h(\zeta)|^2 \log \frac{4}{|\xi - \zeta|^2} \leq \pi \int_{h(B_{r+1})} K(z, f) d\sigma(z)$$

$$\leq \pi \int_{B_{r+3}} K(z, f) d\sigma(z),$$

as desired.

Concerning estimate (2.4), it is equivalent to show that $d(z, \infty) \leq 10/(1 + |f(z)|)$ for all $z \in \widehat{\mathbb{C}}$. If $|z| > 1$ we use (2.6) to obtain

$$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}} \leq \frac{5\sqrt{2}}{1 + 4|z|} \leq \frac{10}{1 + |f(z)|}.$$ 

For $|z| < 1$, in view of the maximum principle $|f(z)| \leq \max_{|\xi| = 1} |f(\xi)| \leq 4$, we conclude

$$d(z, \infty) \leq 2 \leq \frac{10}{1 + |f(z)|}.$$ 

This completes the proof of Proposition 2.2.

3. Proof of Theorem 1

We may assume that $\Omega$ is a subdomain of the unit disk, $\Omega \subset \mathbb{B}$, and $f \neq$ constant. Consider the complex dilatation $\mu = \mu(z)$ of $f$; that is,

$$f_z = \mu(z)f_z \quad \text{a.e. in } \Omega.$$
We extend \( \mu \) by zero outside \( \Omega \) and regard it as a function on the whole of \( \mathbb{C} \). For \( 0 < \varepsilon < 1 \) we define

\[
\mu^\varepsilon(z) = \begin{cases} 
\mu(z) & \text{if } |\mu(z)| \leq 1 - \varepsilon, \\
(1 - \varepsilon)\mu(z)|\mu(z)|^{-1} & \text{if } |\mu(z)| > 1 - \varepsilon.
\end{cases}
\]

Let \( f^\varepsilon: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be the normal solution of the Beltrami equation

\[
f^\varepsilon_z = \mu^\varepsilon(z)f^\varepsilon_z,
\]

and let \( h^\varepsilon = (f^\varepsilon)^{-1}: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) denote the inverse mapping. With the aid of Lemma 1.1 we find that

\[
f^\varepsilon_z = -J(z, f^\varepsilon)h^\varepsilon_\omega \quad \text{and} \quad f^\varepsilon_\omega = J(z, f^\varepsilon)\overline{h^\varepsilon_\omega}.
\]

Therefore, (3.3) becomes a quasi-linear equation for \( h^\varepsilon \) (the hodograph transformation)

\[
h^\varepsilon_\omega = -\mu^\varepsilon(h^\varepsilon(\omega))\overline{h^\varepsilon_\omega}.
\]

The dilatation function of \( f^\varepsilon \) can be estimated independently of \( \varepsilon \) as

\[
K(z, f^\varepsilon) = \frac{1 + |\mu^\varepsilon(z)|}{1 - |\mu^\varepsilon(z)|} \leq \frac{1 + |\mu(z)|}{1 - |\mu(z)|} = K(z, f),
\]

where, in view of our convention, \( K(z, f) \equiv 1 \) outside \( \Omega \). This, combined with Proposition 2.2, leads to uniform estimates

\[
|h^\varepsilon(\xi) - h^\varepsilon(\zeta)|^2 \leq \frac{\pi}{2\log(2/|\xi - \zeta|)} \int_{B_{r+3}} K(z, f) \, d\sigma(z)
\]

for all \( \xi, \zeta \in B_r, \ r > 1 \), with \( |\xi - \zeta| < 2 \), and

\[
d(h^\varepsilon(\xi), \infty) \leq \frac{10}{1 + |\xi|}.
\]

A consequence of (3.5) is that the homeomorphisms \( h^\varepsilon: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) form an equicontinuous family on each compact subset of \( \mathbb{C} \). By the Arzelà-Ascoli theorem it is possible to extract a sequence \( h^{\varepsilon_i}, \ i = 1, 2, \ldots, \varepsilon_i \searrow 0 \), that converges \( \varepsilon \)-uniformly to a mapping \( h: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). Actually, in view of (3.6), the mappings \( h^{\varepsilon_i} \) converge uniformly on the extended complex plane \( \hat{\mathbb{C}} \) with respect to the chordal metric. According to Lemma 1.4 the limit mapping \( h: \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is monotone. In particular, the set

\[
\Omega' = h^{-1}(\Omega)
\]

is a domain.

Other uniform estimates follow from (2.2) and (2.5), particularly,

\[
\int_{B_{r+3}} |Dh^\varepsilon(\omega)|^2 \, d\sigma(\omega) \leq \int_{h(B_{r+1})} K(z, f^\varepsilon) \, d\sigma(z) \leq \int_{B_{r+3}} K(z, f) \, d\sigma(z)
\]

for all \( r \geq 1 \). This shows that \( h^{\varepsilon_i} \) converges weakly in \( W^{1,2}(B_{r+1}) \). Thus \( h \in W^{1,2}_{\text{loc}}(\mathbb{C}) \).

Now we define the function \( \varphi: \Omega' \to \mathbb{C} \) by the rule

\[
\varphi(\omega) = f(h(\omega)) \quad \text{for} \ \omega \in \Omega'.
\]
We want to prove that $\varphi$ is holomorphic. To this end, fix an arbitrary open subset $U \subset \Omega'$, compactly contained in $\Omega'$. Thus $h^{\varepsilon_i}(U) \subset \Omega$ for sufficiently small $\varepsilon_i$, and we can examine the mappings $\varphi^{\varepsilon_i} : U \to \mathbb{C}$ for $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \ldots\}$:

\[ \varphi^{\varepsilon}(\omega) = f(h^{\varepsilon}(\omega)) \quad \text{for} \quad \omega \in U. \]

Applying the chain rule (see Lemma 1.1) we find that $\varphi^{\varepsilon} \in W^{1,2}(U)$ and

\[ \frac{\partial \varphi^{\varepsilon}}{\partial \omega} = f_z h^{\varepsilon}_z + f_{\overline{z}} h^{\varepsilon}_{\overline{z}}, \quad \frac{\partial \varphi^{\varepsilon}}{\partial \overline{\omega}} = f_z h^{\varepsilon}_z + f_{\overline{z}} h^{\varepsilon}_{\overline{z}}. \]

Then, equations (3.1) and (3.4) imply

\[ \varphi^{\varepsilon}(z) = (\mu(z) - \mu^{\varepsilon}(z)) f_z h^{\varepsilon}_z, \]
\[ \varphi^{\varepsilon}(z) = (1 - \mu(z) \mu^{\varepsilon}(z)) f_z h^{\varepsilon}_z, \]

where $z = h^{\varepsilon}(\omega)$. It follows from the definition of $\mu^{\varepsilon}(z)$ (see formula (3.2)) that

\[ |\mu(z) - \mu^{\varepsilon}(z)|^2 \leq \varepsilon (1 - |\mu^{\varepsilon}(z)|^2), \]
\[ |1 - \mu(z) \mu^{\varepsilon}(z)|^2 \leq 1 - |\mu^{\varepsilon}(z)|^2. \]

Notice, too, that $J(\omega, h^{\varepsilon}) = (1 - |\mu^{\varepsilon}|^2)|h^{\varepsilon}_z|^2$ (see (3.4)).

Now we use the change of variables according to Lemma 1.2 to obtain

\[ \int_{U} |\varphi^{\varepsilon}_z|^2 \leq \varepsilon \int_{U} J(\omega, h^{\varepsilon}) |f_z(h^{\varepsilon}(\omega))|^2 d\sigma(\omega) \]
\[ = \varepsilon \int_{h^{\varepsilon}(U)} |f_z(z)|^2 d\sigma(z) \leq \varepsilon \int_{\Omega} |f_z(z)|^2 d\sigma(z). \]

In much the same way we obtain the estimate

\[ \int_{U} |\varphi^{\varepsilon}_{\overline{z}}|^2 d\sigma(\omega) \leq \int_{\Omega} |f_{\overline{z}}(z)|^2 d\sigma(z). \]

These two estimates imply that the sequence $\varphi^{\varepsilon_i} = f(h^{\varepsilon_i})$ converges to the mapping $\varphi = f(h)$ not only pointwise (because $f$ is continuous) but also weakly in $W^{1,2}(U)$. In conclusion, $\varphi \in W^{1,2}(U)$, and we have

\[ \frac{\partial \varphi}{\partial \omega} = 0, \quad \int_{U} |\varphi_z|^2 d\sigma(\omega) \leq \int_{\Omega} |f_z|^2 d\sigma(z). \]

By the Weyl lemma $\varphi$ is holomorphic in $U$. Since $U$ was an arbitrary compact subdomain of $\Omega'$, $\varphi$ is holomorphic in $\Omega'$ as well. This also implies inequality (0.6). To derive inequality (0.5) we use Lemma 1.3

\[ \int_{U} |Dh|^2 d\sigma(\omega) \leq \lim_{\varepsilon \to 0} \int_{U} |Dh^{\varepsilon}|^2 d\sigma(\omega) \leq \lim_{\varepsilon \to 0} \int_{h^{\varepsilon}(U)} K(z, f^{\varepsilon}) d\sigma(z) \]
\[ \leq \int_{\Omega} K(z, f) d\sigma(z). \]

Of course, $U$ can now be replaced by $\Omega'$, proving (0.5).

What remains is to show that $h : \Omega' \to \Omega$ is a homeomorphism, which is the same as to show that $h$ is one-to-one.
Recall that our function \( f \in W^{1,2}(\Omega) \) is actually continuous and nonconstant. For a given point \( a \in \Omega \), its preimage \( h^{-1}(a) \subset \Omega' \) is a continuum (compact connected set) because \( h \) is a monotone mapping. Clearly, the analytic function \( \varphi = f \circ h \) is constant on \( h^{-1}(a) \). Hence \( h^{-1}(a) \) consists of a single point, because otherwise \( \varphi \) would be constant on the whole of \( \Omega' \); thus, \( f \) would be constant on \( \Omega \).

In conclusion, \( h : \Omega' \to \Omega \) is a homeomorphism, and we have the factorization \( f = \varphi \circ h^{-1} \).

This proves Theorem 1.

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