ON COMPOSITIONS OF CONFORMAL IMMERSIONS

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Abstract. We consider conformal immersions of a manifold $M^n$, $n \geq 6$, into conformally flat manifolds. If the principal curvatures of $f : M^n \to \widetilde{N}^{n+1}_{cf}$ have multiplicities at most $n-4$, we show that any $g : M^n \to \widetilde{N}^{n+2}_{cf}$ can locally be written as $g = \rho \circ f$, where $\rho : \widetilde{N}^{n+1}_{cf} \to \widetilde{N}^{n+2}_{cf}$ is a conformal immersion.

1. Introduction

A classical result due to Cartan [Ca] states that a codimension one conformal immersion $f : M^n \to \widetilde{N}^{n+1}_{cf}$ of an $n$-dimensional Riemannian manifold into a conformally flat Riemannian manifold is (locally) conformally rigid if $n \geq 5$ and the maximal multiplicity of the principal curvatures satisfies $\nu^c_f \leq n - 3$ everywhere. Then any other conformal immersion $g : M^n \to \widetilde{N}^{n+1}_{cf}$ is locally a composition $g = \rho \circ f$ for some local conformal diffeomorphism $\rho : \widetilde{N}^{n+1}_{cf} \to \widetilde{N}^{n+1}_{cf}$. Cartan's result was extended to codimension greater than one in [dCD].

For fixed $k \geq 2$, a natural problem is to find conditions on $f : M^n \to \widetilde{N}^{n+1}_{cf}$ which imply that any conformal immersion $g : M^n \to \widetilde{N}^{n+k}_{cf}$ is locally a conformal composition. That $g$ is a local conformal composition means that, for each point $x \in M^n$, there exists a neighborhood $V \subset M$ of $x$ and a conformal immersion $\rho : W \subset \widetilde{N}^{n+1}_{cf} \to \widetilde{N}^{n+k}_{cf}$ of an open subset of $\widetilde{N}^{n+1}_{cf}$ containing $f(V)$ such that $g = \rho \circ f$ along $V$. When $k = 2$, we prove the following result.

Theorem 1. Let $f : M^n \to \widetilde{N}^{n+1}_{cf}$ be a conformal immersion. Assume that $n \geq 6$ and $\nu^c_f(x) \leq n - 4$ everywhere. If $g : M^n \to \widetilde{N}^{n+2}_{cf}$ is a conformal immersion then there exists an open dense subset $\mathcal{U} \subset M$ such that, when restricted to $\mathcal{U}$, $g$ is a local conformal composition.

2. The proof

We say that a submanifold $\overline{N}^{n+1}_{cf} \subset \widetilde{N}^{n+2}_{cf}$ is a conformally flat hypersurface if, with the metric induced by the inclusion map, $\overline{N}^{n+1}_{cf}$ is conformally flat. Using Cartan's result, it is easy to check that Theorem 1 is equivalent to the following:
Theorem 2. Let \( f: M^n \to N^{n+1}_{cf} \) be a conformal immersion. Assume that \( n \geq 6 \) and \( \nu_2^c(x) \leq n-4 \) everywhere. If \( g: M^n \to \tilde{N}^{n+2}_{cf} \) is a conformal immersion then there exists an open dense subset \( \mathcal{U} \subset M^n \) such that \( g|_\mathcal{U} \) is locally contained in a conformally flat hypersurface of \( \tilde{N}^{n+2}_{cf} \).

To prove Theorem 2 we will make use of the following lemma on flat bilinear forms. We refer the reader to [dCD] or [Da] for notation, definitions, and some basic facts.

Lemma 3. Let \( \beta: V \times V \to W^{k,2}, \ k \geq 3, \) be a nonzero symmetric bilinear form. Assume that \( \beta \) is flat and \( \dim N(\beta) < \dim V - \dim W. \) Then \( W \) admits an orthogonal direct sum decomposition \( W = W_1^{r,2} \oplus W_2^{k-r,2-r}, \) where \( r = 1 \) or 2, such that if \( \beta_1 \) and \( \beta_2 \) are the \( W_1 \) and \( W_2 \) components of \( \beta \), respectively, then

(i) \( \beta_1 \) is nonzero and null,
(ii) \( \beta_2 \) is flat and \( \dim N(\beta_2) \geq \dim V - \dim W_2. \)

Proof. Analogous to that of Lemma 2.2 in [dCD]. □

Proof of Theorem 2. We may assume that \( N^{n+1}_{cf} = S^{n+1} \) is the unit Euclidean sphere, that \( \tilde{N}^{n+2}_{cf} = \mathbb{R}^{n+2} \) is the flat Euclidean space, and that \( M^n \) is endowed with the metric induced by \( g. \) We consider \( S^{n+1} \) isometrically embedded in the light-cone \( V^{n+2} \) of the flat Lorentzian space \( L^{n+3} \) and contained in an \( (n+2) \)-dimensional affine hyperplane orthogonal to the axis of \( V^{n+2}. \)

The map \( F: M^n \to V^{n+2} \subset L^{n+3} \) defined by

\[
F(x) = \frac{1}{\varphi(x)} f(x)
\]

is an isometric immersion, where \( \varphi: M^n \to \mathbb{R} \) is the positive function satisfying

\[
\langle f_*(x)X, f_*(x)Y \rangle = \varphi^2(x) \langle X, Y \rangle
\]

for any \( X, Y \in T_xM. \)

As in [dCD] or [Da], for a fixed point \( x \in M^n, \) the vector-valued second fundamental form \( \alpha_F: TM \times TM \to T_F M^\perp \) of \( F \) in \( L^{n+3} \) is given by

\[
\alpha_F = (\langle \alpha_F, \eta \rangle + \langle , \rangle)\xi + \langle \alpha_F, \eta \rangle \eta + \alpha_F^*,
\]

where the basis \( \xi, \eta \) for the orthogonal complement of \( T_{f(x)}M^\perp \) into \( T_{f(x)}M^\perp \) verifies

\[
\langle \xi, \xi \rangle = 1, \quad \langle \xi, \eta \rangle = 0, \quad \langle \eta, \eta \rangle = -1
\]

and \( F(x) = \xi + \eta. \) Here \( \alpha_F^* \) is the \( T_{f(x)}M^\perp \) component of \( \alpha_F \) and satisfies

\[
(1) \quad \alpha_F^* = \alpha_F/\varphi.
\]

Now let

\[
W = T_{g(x)}M^\perp \oplus \text{Span}\{\xi\} \oplus \text{Span}\{\eta\} \oplus T_{f(x)}M^\perp
\]

be given the natural metric \( \langle , \rangle \) of type \( (3,2). \) Define \( \beta: T_x M \times T_x M \to W \) by

\[
\beta = \alpha_g + \langle \alpha_F, \eta \rangle + \langle , \rangle)\xi + \langle \alpha_F, \eta \rangle \eta \oplus \alpha_F^*.
\]
A straightforward computation shows that
\[ \langle \beta(X, Y), \beta(Z, W) \rangle - \langle \beta(X, W), \beta(Z, Y) \rangle = \langle \alpha_g(X, Y), \alpha_g(Z, W) \rangle - \langle \alpha_g(X, W), \alpha_g(Z, Y) \rangle \]
\[ - \langle \alpha_F(X, Y), \alpha_F(Z, W) \rangle + \langle \alpha_F(X, W), \alpha_F(Z, Y) \rangle, \]
and the Gauss equations for \( g \) and \( F \) imply that \( \beta \) is flat.

By definition of \( \beta \), we have \( \beta(X, X) \neq 0 \) for \( X \neq 0 \); thus, \( N(\beta) = 0 \). By Lemma 3, \( W = W_1 \oplus W_2 \) decomposes orthogonally so that \( \beta = \beta_1 \oplus \beta_2 \), where
\[ \beta_1: T_xM \times T_xM \to W_1^{r-1}, \quad r \in \{1, 2\}, \]
is nonzero and null and
\[ \beta_2: T_xM \times T_xM \to W_2^{3-r, 2-r} \]
is flat satisfying \( \dim N(\beta_2) \geq n - 5 + 2r \).

We claim that \( r = 2 \). Assume \( r = 1 \). It follows that \( \beta_1 = \phi \gamma \), where \( \gamma \in W_1 \) is a null vector and \( \phi \) is a real-valued symmetric bilinear form. Thus there exists a unit vector \( \delta \in T_{g(x)}M^\perp \) such that
\[ \gamma = \cos \theta \delta + \sin \theta \xi + \cos \theta \eta + \sin \theta N, \]
where \( N \in T_{f(x)}M^\perp \) is a unit vector. By definition, we have \( Z \in N(\beta_2) \) if and only if \( \beta(Z, X) = \beta_1(Z, X) = \phi(Z, X) \gamma \) for all \( X \in T_xM \); therefore,

(2) \[ \langle \alpha_F(Z, X), \eta \rangle \]
(3) \[ \langle \alpha_F(Z, X), \eta \rangle = \phi(Z, X) \cos \theta, \]
and

(4) \[ \langle \alpha_F^*(Z, X), N \rangle = \phi(Z, X) \sin \theta \]
for all \( Z \in N(\beta_2) \) and \( X \in T_xM \). From (2) and (3) we get

(5) \[ \phi(Z, X)(\sin \theta - \cos \theta) = \langle Z, X \rangle, \]
which implies \( \sin \theta - \cos \theta \neq 0 \). From (4) and (5) we obtain

(6) \[ \langle \alpha_F^*(Z, X), N \rangle = \frac{\sin \theta}{\sin \theta - \cos \theta} \langle Z, X \rangle. \]

Using (1), we conclude from (6) that \( f \) has a principal curvature with multiplicity at least \( \dim N(\beta_2) \geq n - 3 \). This is a contradiction and proves the claim.

Since \( r = 2 \), we have \( \beta_1 = \phi_1 \gamma_1 + \phi_2 \gamma_2 \), where \( \phi_1, \phi_2 \) are real-valued symmetric bilinear forms and \( \gamma_1, \gamma_2 \) are orthogonal null vectors. So we may write

(7) \[ \gamma_1 = \eta + \cos u \xi + \sin u \delta_1 \]
and

(8) \[ \gamma_2 = N + \cos v \xi + \sin v \delta_2, \]
where \( \delta_1, \delta_2 \) are unit vectors in \( T_{g(x)}M^\perp \) verifying
\[ \cos u \cos v + \sin u \sin v \langle \delta_1, \delta_2 \rangle = 0. \]
Clearly, $\phi_1 = \langle \alpha_F, \eta \rangle$ and $\phi_2 = \langle \alpha_F^n, N \rangle$. Hence,

$$
\beta_1 = \langle \alpha_F, \eta \rangle (\eta \cos u \xi + \sin u \delta_1) + \langle \alpha_F^n, N \rangle (N \cos u \xi + \sin u \delta_2).
$$

For any $Z \in N(\beta_2)$ and $X \in T_x M$, $\beta(Z, X) = \beta_1(Z, X)$ is equivalent to

$$
\alpha_g(Z, X) = \langle \alpha_F(Z, X), \eta \rangle \sin u \delta_1 + \langle \alpha_F^n(Z, X), N \rangle \sin u \delta_2
$$

and

$$
\langle \alpha_F(Z, X), \eta \rangle (1 - \cos u) + \langle Z, X \rangle = \langle \alpha_F^n(Z, X), N \rangle \cos v.
$$

Thus, from $v \leq n - 4$, we have $1 - \cos u \neq 0$ and $\cos v \neq 0$; therefore,

$$
\alpha_g(Z, X) = \langle \alpha_F(Z, X), \eta \rangle (\sin u \delta_1 + \tan v (1 - \cos u) \delta_2) + \tan v \langle Z, X \rangle \delta_2
$$

and

$$
\alpha_g(Z, X) = \langle \alpha_F^n(Z, X), N \rangle \left( \frac{\sin u \cos v}{1 - \cos u} \delta_1 + \sin u \delta_2 \right)
$$

$$
- \frac{\sin u}{1 - \cos u} \langle Z, X \rangle \delta_1.
$$

We easily conclude from (10) that $g$ has a normal direction $\sigma$ such that the tangent-valued second fundamental form $A_\sigma$ in this direction has an eigenvalue with multiplicity at least $n - 1$ whose eigenspace contains $N(\beta_2)$.

From $r = 2$ we have that $\dim S(\beta) = 2, 3$. We claim that $\dim S(\beta) = 2$ if and only if $\sigma$ is an umbilical direction. First observe that $\dim S(\beta) = 2$ if and only if $\beta_2 = 0$, and if $\beta_2 = 0$ then $\sigma$ is umbilical by equation (10). Conversely, if $A_\sigma = c I$, consider the vector $\zeta = \sigma / c - \xi - \eta$. Then $\zeta$ is not null and $\langle \langle \beta, \zeta \rangle \rangle = 0$. This implies that $\dim S(\beta) = 2$ and proves the claim.

Assume that $\dim S(\beta) = 3$ on an open subset $V \subset M^n$. The 2-dimensional distribution $S(\beta) \cap S(\beta \perp)$ is the (maximal) degeneracy space of the restriction of $\langle \langle , \rangle \rangle$ to the smooth distribution $S(\beta)$ and, therefore, is smooth. It follows easily that the vector fields $\delta_1, \delta_2$ and the functions $u, v$ in (7) and (8) can be taken to be smooth on $V$. The same conclusion holds on any open subset of $M$ where $\dim S(\beta) = 2$.

Let $W \subset M$ be the open subset of points where $\dim S(\beta) = 3$, and let $W_1$ be the interior of $M \setminus W$. Let $\sigma$ be a smooth umbilical unit normal vector field defined on a connected component $U_\lambda$ of $W_1$. We claim that $\sigma$ is parallel with respect to the normal connection of $g$. In fact, if $\sigma$ is not parallel at $x \in U_\lambda$, we easily conclude from the Codazzi equation for $A_\sigma$ that the second fundamental form $A_\sigma$ has a principal curvature with multiplicity at least $n - 1$. The same holds in a neighborhood $W \subset U_\lambda$ of $x$, and it is a well-known fact that $W$ must be conformally flat (cf. [CY]). By the classical Cartan-Schouten theorem for conformally flat hypersurfaces, we conclude that $v/\kappa \geq n - 1$ on $W$, which is a contradiction and proves the claim. It follows from the claim that $g(U_\lambda)$ is contained in an umbilical hypersurface of $R^{n+2}$.

For a connected component $V_1$ of $W$, let $\sigma$ be a smooth unit normal vector field such that the second fundamental form $A_\sigma$ has eigenvalues $\mu, \lambda$ with multiplicities $1$ and $(n - 1)$, respectively. Set $\Delta = \ker(A_\sigma - \lambda I)$. We claim that $\lambda$ is constant and $\sigma$ is parallel along $\Delta$. Consider orthonormal vector fields $Y_1, \ldots, Y_{n-1} \in \Delta$ such that $\nabla_{Y_j}^2 \sigma = 0$ for $2 \leq j \leq n - 1$. From the
Codazzi equation for $A_\sigma$, $Y_i$, and $Y_j$ for $2 \leq i \neq j \leq n - 1$, we easily conclude that

$$Y_j(\lambda) = 0, \quad 2 \leq j \leq n - 2.$$  

Now the Codazzi equation for $A_\sigma$, $Y_1$, and $Y_j$ yields

$$\langle \nabla_{Y_1}^\perp \sigma, \sigma^\perp \rangle \langle A_\sigma \cdot Y_1, Y_j \rangle = 0, \quad 1 \leq i \neq j \leq n - 1,$$

and

$$Y_1(\lambda) = \langle \nabla_{Y_1}^\perp \sigma, \sigma^\perp \rangle \langle A_\sigma \cdot Y_j, Y_j \rangle, \quad 2 \leq j \leq n - 1.$$  

If at some point $\langle \nabla_{Y_1}^\perp \sigma, \sigma^\perp \rangle \neq 0$, we obtain from (12) and (13) that $\text{Span}\{Y_2, \ldots, Y_{n-1}\}$ contains an $(n - 3)$-dimensional umbilical subspace for $g$. Now (1) and (11) imply that $\nu_f(x) \geq n - 3$, which is not possible, and this proves the claim.

Set $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$, where $\mathcal{U}_3 \subset \mathcal{W}$ is the open subset where $\lambda \neq 0$ and $\mathcal{U}_2$ is the interior of $\mathcal{W} \setminus \mathcal{U}_3$.

The image under $g$ of any connected component of $\mathcal{U}_2$ is contained in a flat hypersurface of $\mathbb{R}^{n+2}$ by Proposition 3 of [DG]. If $V_\lambda$ is a connected component of $\mathcal{U}_3$, define $c: V_\lambda \to \mathbb{R}^3$ by

$$c(x) = g(x) + r(x)\sigma(x), \quad r(x) = 1/\lambda(x).$$

For all $Y \in \Delta$, we have

$$\tilde{\nabla}_Y c = Y - rA_\sigma Y = 0,$$

where $\tilde{\nabla}$ denotes the canonical connection of $\mathbb{R}^{n+2}$. If $X$ is a unit tangent vector field orthogonal to $\Delta$, we get

$$\tilde{\nabla}_X c = X - X(r)\sigma - rA_\sigma X - r\nabla_X^\perp \sigma.$$  

In particular, since $\sigma$ is not an umbilical direction, we have

$$||\tilde{\nabla}_X c||^2 > |X(r)|^2;$$

hence, from the curve $c$ and the function $r$ we can construct a conformally flat hypersurface in $\mathbb{R}^{n+2}$ as described in [dCDM] or [Da] which contains $g(V_\lambda)$. This concludes the proof. □

References


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