FACTORIZATION OF GENERIC MAPPINGS BETWEEN SURFACES

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Abstract. Given a generic mapping $F : S \rightarrow N$ of two smooth (i.e., $C^\infty$) real surfaces, $S$ compact, and a line bundle $\pi : E \rightarrow N$, we look for necessary and sufficient conditions to find an immersion $\tilde{F} : S \rightarrow E$ such that $F = \pi \circ \tilde{F}$.

0. Introduction

Let $S, N$ be two differentiable surfaces (i.e., real, $C^\infty$, 2-manifolds), $S$ compact, and let $F : S \rightarrow N$ be a generic mapping, that is, a mapping locally equivalent to one of the following:

(i) $(x, y) \mapsto (x, y)$,
(ii) $(x, y) \mapsto (x, y^2)$,
(iii) $(x, y) \mapsto (x, y^3 - xy)$,

and whose apparent contour is a smooth curve except for a finite number of normal crossings and semicubical cusps (compare [6, 1]).

Let $\pi : E \rightarrow N$ be a differentiable line bundle (i.e., a rank 1 vector bundle). We shall answer the following:

Question. Does there exist an immersion (i.e., a mapping with injective differential at every point) $\tilde{F} : S \rightarrow E$ such that $F = \pi \circ \tilde{F}$? (i.e., the following is a commutative diagram):

\[ \begin{array}{ccc} & E & \\ & \downarrow \pi & \\ \tilde{F} & S \rightarrow N & F \\ \end{array} \]

This question was first answered by Haefliger [2], in the case $N = \mathbb{R}^2$ and $E = \mathbb{R}^3$, and his theorem and proof were later generalized by Millet [3] to the case of an arbitrary surface $N$ and $E = N \times \mathbb{R}$.

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In this paper we use again Haefliger's original idea to deal with the general case. In the first section, we state Theorem 1.1, which answers the question; in the second we give the proof of this theorem; and finally in the third section we apply it to the problem of finding a factorization of a generic mapping \( F : S \to \mathbb{RP}^2 \) by means of an immersion in \( \mathbb{RP}^3 \) and a projection from a point.

1. Statement of the theorem

Let \( \Sigma \) denote the set of critical points of \( F \), \( C \) a connected component of \( \Sigma \), and \( f_C \) the restriction of \( F \) to \( C \). We can define the following two line bundles over the base space \( C \):

1. \( \kappa_C : K_C \to C \) the bundle of kernels of \( dF \) (i.e., \( \kappa_C^{-1}(p) = \ker(dF(p)) \forall p \in C \));
2. \( f_C^*\pi : f^*E \to C \) the induced bundle.

We shall prove the following.

**Theorem 1.1.** There exists an immersion \( \tilde{F} : S \to E \) such that \( F = \pi \circ \tilde{F} \) if and only if for all the components \( C \) of \( \Sigma \) the Whitney sum of the previous two bundles is trivial.

**Remark.** Let \( \zeta : Z \to C \) be a line bundle over \( C \), and define

\[
\varepsilon(\zeta) = \begin{cases} 
1 & \text{if } \zeta \text{ is orientable}, \\
-1 & \text{if } \zeta \text{ is nonorientable}.
\end{cases}
\]

(In some sense this number is the Stiefel-Whitney class of the bundle.)

It is easily seen that the condition in Theorem 1.1 is equivalent to

\[
\varepsilon(\kappa_C)\varepsilon(f_C^*\pi) = 1;
\]

that is, either both bundles are orientable or both are nonorientable.

Furthermore, let \( c(C) \) denote the number of cusp points in \( C \), and \( \nu_C : N_C \to C \) the normal bundle of \( C \) in \( S \). By [2, Lemma 1], we get

\[
\varepsilon(\kappa_C) = (-1)^c(C)\varepsilon(\nu_C);
\]

so condition (1.1) turns into

\[
(-1)^c(C)\varepsilon(\nu_C)\varepsilon(f_C^*\pi) = 1.
\]

Finally observe that, if \( \pi : E \to N \) is the trivial bundle, then \( \varepsilon(f_C^*\pi) = 1 \) and thus (1.2) reduces to Haefliger-Millet's condition.

2. Proof of the theorem

First of all observe that finding a mapping \( \tilde{F} : S \to E \) such that \( \pi \circ \tilde{F} = F \) is, by the very definition of the induced bundle, the same as finding a cross-section \( \sigma \) of the bundle \( F^*\pi : F^*E \to S \), induced from \( \pi : E \to N \) (see the diagram):

\[
\begin{array}{ccc}
F^*E & \xrightarrow{\pi^*F} & E \\
\sigma \uparrow & \downarrow F^*\pi & \downarrow \pi \\
S & \xrightarrow{F} & N
\end{array}
\]
Now, if \( \sigma \) is a section of \( F^*\pi : F^*E \to S \), then
\[
\forall p \in S - \Sigma \quad d(\pi^*F \circ \sigma)(p) \text{ is injective},
\]

since a section is always an immersion and the set of critical points of \( \pi^*F \) is \( (F^*\pi)^{-1}(\Sigma) \). (Roughly speaking, the obstruction to making \( \pi^*F \circ \sigma \) an immersion is "concentrated" around \( \Sigma \).) This means that it suffices to find a section \( \sigma_1 \) of the bundle \( F^*E_{|V} \to V \) being a tubular neighborhood of \( \Sigma \)—such that \( \pi^*F \circ \sigma_1 \) is an immersion; an arbitrary extension of this section—which can always be found—will provide the desired immersion.

Since \( V \) is a tubular neighborhood of \( \Sigma \), it has the same number of connected components as \( \Sigma \). Let \( U \) be the connected component of \( V \) containing \( C \).

\( U \cong \nu_C \) is diffeomorphic to the quotient
\[
\mathbb{R}^2/(x,y)\sim(x+1,y) \quad \text{if } e(\nu_C) = 1,
\]
\[
\mathbb{R}^2/(x,y)\sim(x+1,-y) \quad \text{if } e(\nu_C) = -1,
\]
in such a way that the curve \( C \) is mapped onto the quotient of the line \( \{y = 0\} \).

Then the bundle \( F^*\pi : F^*E_{|U} \to U \) is isomorphic to one of the following four:

(I) \( \mathbb{R}^3/(x,y,z)\sim(x+1,y,z) \to \mathbb{R}^2/(x,y)\sim(x+1,y) \) if \( e(\nu_C) = 1 \) and \( e(f_C^*\pi) = 1 \);

(II) \( \mathbb{R}^3/(x,y,z)\sim(x+1,y,-z) \to \mathbb{R}^2/(x,y)\sim(x+1,-y) \) if \( e(\nu_C) = -1 \) and \( e(f_C^*\pi) = 1 \);

(III) \( \mathbb{R}^3/(x,y,z)\sim(x+1,-y,z) \to \mathbb{R}^2/(x,y)\sim(x+1,-y) \) if \( e(\nu_C) = 1 \) and \( e(f_C^*\pi) = -1 \);

(IV) \( \mathbb{R}^3/(x,y,z)\sim(x+1,-y,-z) \to \mathbb{R}^2/(x,y)\sim(x+1,y) \) if \( e(\nu_C) = -1 \) and \( e(f_C^*\pi) = -1 \);

as follows from the fact that \( U \) deforms onto \( C \) and the Lifting Homotopy Theorem for fibre bundles (compare [5]), where each \( \pi_i \) denotes the mapping induced by the canonical projection \((x, y, z) \mapsto (x, y)\).

**Lemma 2.1.** Every cross-section of the line bundle \( (I) \) [resp. \( (II), (III), (IV) \)] defines a function \( h : \mathbb{R}^2 \to \mathbb{R} \) such that \( (I) \) [resp. \( (II), (III), (IV) \)] holds:

(I) \( h(x + 1, y) = h(x, y) \) \( \forall x, y \);

(II) \( h(x + 1, -y) = h(x, y) \) \( \forall x, y \);

(III) \( h(x + 1, y) = -h(x, y) \) \( \forall x, y \);

(IV) \( h(x + 1, -y) = -h(x, y) \) \( \forall x, y \).

Conversely every such function defines a cross-section of the corresponding bundle.

**Proof.** Obvious. \( \square \)

Let \( K_x \) denote the fiber of the bundle \( \kappa : K \to C \) over \((x, 0)\) (i.e., the kernel of \( dF(x, 0) \)).

**Lemma 2.2.** Let \( h \) be a function as in the previous lemma. Then \( h \) defines a cross-section \( \sigma_1 \) making \( \pi^*F \circ \sigma_1 \) an immersion if and only if
\[
(2.1) \quad \forall x \in \mathbb{R}, \forall (u, v) \in K_x - \{(0, 0)\} \quad \frac{\partial h}{\partial x}(x, 0)u + \frac{\partial h}{\partial y}(x, 0)v \neq 0.
\]

**Proof.** \( \pi^*F \circ \sigma_1 \) is an immersion if and only if
\[
\forall x \in \mathbb{R} \quad d\sigma_1(x, 0)[K_x] \not\subseteq \ker(d(\pi^*F)(\sigma_1(x, 0))),
\]

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but with our notation

\[ d\sigma_1(x, 0)[K_x] = \left\{ (u, v, \frac{\partial h}{\partial x}(x, 0)u + \frac{\partial h}{\partial y}(x, 0)v) \mid (u, v) \in K_x \right\}; \]

\[ \ker(d(\pi^*F)(x, 0, z)) = \{(u, v, w) \mid (u, v) \in K_x, w = 0\}; \]

thus the thesis holds. \( \square \)

**Lemma 2.3.** Use the coordinates on \( U \cong \nu_C \) given before Lemma 2.1. The line bundle \( K \) is orientable (i.e., \( \varepsilon(\nu_C) = 1 \)) if and only if there exists a never zero function \( k : \mathbb{R} \to \mathbb{R}^2, \quad k(x) = (k_1(x), k_2(x)), \) such that

\[ (2.2) \quad \forall x \in \mathbb{R} \quad dF(x, 0)[k(x)] = 0; \]

\[ \forall x \in \mathbb{R} \quad k_1(x + 1) = k_1(x) \quad \text{and} \quad k_2(x + 1) = \begin{cases} k_2(x) & \text{if } \varepsilon(\nu_C) = 1, \\ -k_2(x) & \text{if } \varepsilon(\nu_C) = -1. \end{cases} \]

On the contrary, \( K \) is nonorientable (i.e., \( \varepsilon(\nu_C) = -1 \)) if and only if there exists a never zero function \( k : \mathbb{R} \to \mathbb{R}^2, \quad k(x) = (k_1(x), k_2(x)), \) such that

\[ (2.3) \quad \forall x \in \mathbb{R} \quad dF(x, 0)[k(x)] = 0; \]

\[ \forall x \in \mathbb{R} \quad k_1(x + 1) = -k_1(x) \quad \text{and} \quad k_2(x + 1) = \begin{cases} -k_2(x) & \text{if } \varepsilon(\nu_C) = 1, \\ k_2(x) & \text{if } \varepsilon(\nu_C) = -1. \end{cases} \]

**Proof.** A line bundle over \( C \) is orientable if and only if it has a never zero cross-section, and such a section for the bundle \( K \) is provided by a function \( k \) as in (2.2). The second part of the statement is proved by a similar argument. \( \square \)

By Lemmas 2.1 and 2.2 and the considerations at the beginning of this section, it follows that proving Theorem 1.1 is the same as proving

**Theorem 2.4.** There exists a function \( h \) satisfying condition (I) [resp. (II), (III), (IV)] of Lemma 2.1 and condition (2.1) of Lemma 2.2 if and only if (1.1) holds.

**Proof** ((1.1) is sufficient). There are four possibilities, corresponding to the four bundles (I), (II), (III), (IV).

(I), (II). Since \( \varepsilon(f_C^*\pi) = 1 \), (1.1) implies \( \varepsilon(\kappa) = 1 \), so the assumption and thesis are the same as in [2, Lemma 2], thus the thesis holds.

(III). Since \( \varepsilon(f_C^*\pi) = -1 \), (1.1) implies \( \varepsilon(\kappa) = -1 \). Then by (2.3) we have a never zero function \( k \) such that

\[ (2.4) \quad \forall x \in \mathbb{R} \quad k_1(x + 1) = -k_1(x) \quad \text{and} \quad k_2(x + 1) = -k_2(x). \]

Define

\[ h(x, y) = r(x) + yk_2(x) \]

where

\[ r(x) = \int_0^x k_1(t) \, dt - \frac{1}{2} \int_0^1 k_1(t) \, dt. \]
Observe that
\[
\begin{align*}
  r(x + 1) + r(x) &= \int_0^{x+1} k_1(t) \, dt + \int_0^x k_1(t) \, dt - \int_0^1 k_1(t) \, dt \\
  &= \int_0^x k_1(t) \, dt + \int_1^{x+1} k_1(t) \, dt \\
  &= \int_0^x k_1(t) \, dt + \int_0^x k_1(t + 1) \, dt \quad \text{[by (2.4)]} \\
  &= \int_0^x k_1(t) \, dt - \int_0^x k_1(t) \, dt = 0;
\end{align*}
\]
and using (2.4) again we have
\[
  h(x + 1, y) = -h(x, y) \quad \forall x, y,
\]
that is, condition (III) holds. Furthermore \( \nabla h(x, 0) = k(x) \), so (2.1) holds too.

(IV). Once again (1.1) implies \( \varepsilon(\kappa_C) = -1 \); therefore, by (2.3) we have a never zero function \( k \) such that
\[
(2.5) \quad \forall x \in \mathbb{R} \quad k_1(x + 1) = -k_1(x) \quad \text{and} \quad k_2(x + 1) = k_2(x).
\]

As before define \( h(x, y) = r(x) + y k_2(x) \), where
\[
r(x) = \int_0^x k_1(t) \, dt - \frac{1}{2} \int_0^1 k_1(t) \, dt,
\]
and use (2.5) to get
\[
h(x + 1, -y) = -h(x, y), \quad \nabla h(x, 0) = k(x),
\]
that is, the thesis.

(1.1) is necessary). Suppose that such a function \( h \) exists. Then the projection of \( \nabla h(x, 0) \) on \( K_x \) will provide a never zero function \( k \) such that either (2.2)—in cases (I) or (II)—or (2.3)—in cases (III) or (IV)—holds. This ends the proof of the theorem. \( \square \)

3. Generic mappings in \( \mathbb{R}P^2 \)

Let \( \mathbb{R}P^3 \) denote projective space, and let \( p \in \mathbb{R}P^3 \) be a fixed point. Identify \( \mathbb{R}P^2 \) with the set of lines in \( \mathbb{R}P^3 \) through the point \( p \). There is a canonical projection
\[
\pi : \mathbb{R}P^3 - \{p\} \to \mathbb{R}P^2.
\]

Let \( F : S \to \mathbb{R}P^2 \) be a generic mapping. One can ask for the existence of an immersion \( \tilde{F} : S \to \mathbb{R}P^3 - \{p\} \) such that \( F = \pi \circ \tilde{F} \).

**Proposition 3.1.** With the just said assumptions and notation, such an \( \tilde{F} \) exists if and only if for all connected components \( C \) of the critical set \( \Sigma \) of \( F \)
\[
(-1)^{c(C)} \varepsilon(\nu_C) = 1.
\]

**Remark.** The condition is exactly the same found by Haefliger [2] looking for a factorization with an immersion in \( \mathbb{R}^3 \) of a generic mapping in \( \mathbb{R}^2 \).
Proof. Observe that the projection $\pi : \mathbb{RP}^3 - \{p\} \rightarrow \mathbb{RP}^2$ is nothing but the tautological bundle over $\mathbb{RP}^2$; thus we are allowed to use Theorem 1.1 and the thesis will be proved once we show that $f_C^* \pi : f_C^* E \rightarrow C$ is the trivial bundle for all $C$. But this is a consequence of the following:

**Lemma 3.2.** For all components $C$ of $\Sigma$ the curve $f_C : C \rightarrow \mathbb{RP}^2$ is homotopically trivial.

**Proof.** First of all, observe that the curve $f_C$ is sided, which means it possesses a field of transverse vectors, excepted at cusp points, that in a neighborhood of each cusp is directed towards the internal part of the cusp. Such a field can be defined by the direction toward which the map $F$ folds (see Figure 1).

It is not hard to see that the curve $f_C$ can be deformed, by means of an homotopy, to a regular (i.e., with never zero derivative) sided curve (see Figure 2). Thus, to prove the lemma, it is enough to prove the following:

**Lemma 3.3.** Any regular sided closed curve in the projective plane is homotopically trivial.

**Proof.** Let $f : [0, 1] \rightarrow \mathbb{RP}^2$ be such a curve, that is, 

$$f(1) = f(0), \quad f'(1) = f'(0),$$

and let $n(t)$ be a field of transverse vectors along $f$.

Let $\tilde{f}$ be the lifting of $f$ to the sphere. Then it is easily seen that $f$ is homotopically trivial if and only if $\tilde{f}$ is a closed curve. Suppose, for the sake of contradiction, $\tilde{f}(0) \neq \tilde{f}(1)$, and let $\tilde{n}(t)$ denote the lifting of the vector field $n$.

Since the quotient map from the sphere to the projective plane identifies opposite points and opposite tangent vectors at opposite points, we have that the following hold:

$$(3.1) \quad \tilde{f}(1) = -\tilde{f}(0), \quad \tilde{f}'(1) = -\tilde{f}'(0), \quad \tilde{n}(1) = -\tilde{n}(0).$$

By our assumptions, the three vectors $\tilde{f}(t), \tilde{f}'(t),$ and $\tilde{n}(t)$ are linearly independent for all $t \in [0, 1]$; therefore,

$$\det(\tilde{f}(t) \mid \tilde{f}'(t) \mid \tilde{n}(t)) \neq 0 \quad \forall t \in [0, 1];$$
on the other hand, using (3.1) we have

$$\det(\dot{\tilde{f}}(1) \mid \dot{\tilde{f}}(1) \mid \tilde{n}(1)) = -\det(\dot{f}(0) \mid \dot{f}(0) \mid \tilde{n}(0)),$$

and this is a contradiction.

This ends the proof of the two lemmas and of Proposition 3.1. □

Remark. The converse of Lemma 3.3 is also true. In fact if $f$ is homotopically trivial, then it has a closed lifting $\tilde{f}$ to the sphere. Since the sphere is orientable, $\tilde{f}$ possesses a field $\tilde{n}$ of transverse vectors. The covering map transforms $\tilde{n}$ to a field of transverse vectors along $f$ (see also [4, Proposition 1]).

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