PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 2, June 1993

WEAK-POLYNOMIAL CONVERGENCE ON A BANACH SPACE

J. A. JARAMILLO AND A. PRIETO

(Communicated by Theodore W. Gamelin)

ABSTRACT. We show that any super-reflexive Banach space is a Λ-space (i.e., the weak-polynomial convergence for sequences implies the norm convergence). We introduce the notion of κ-space (i.e., a Banach space where the weak-polynomial convergence for sequences is different from the weak convergence) and we prove that if a dual Banach space Z is a κ-space with the approximation property, then the uniform algebra $A(B)$ on the unit ball of Z generated by the weak-star continuous polynomials is not tight.

We shall be concerned in this note with some questions, posed by Carne, Cole, and Gamelin in [3], involving the weak-polynomial convergence and its relation to the tightness of certain algebras of analytic functions on a Banach space.

Let X be a (real or complex) Banach space. For $m = 1, 2, \ldots$ let $\mathscr{P}(^m X)$ denote the Banach space of all continuous m-homogeneous polynomials on X, endowed with its usual norm; that is, each $P \in \mathscr{P}(^m X)$ is a map of the form $P(x) = A(x, \ldots, ^{m)} x)$, where A is a continuous, m-linear (scalar-valued) form on $X \times \cdots ^{m)} \times X$.

Also, let $\mathscr{P}(X)$ denote the space of all continuous polynomials on X; that is, each $P \in \mathscr{P}(X)$ is a finite sum $P = P_0 + P_1 + \cdots + P_n$, where P_0 is constant and each $P_m \in \mathscr{P}(^m X)$ for $m = 1, 2, \ldots, n$.

In [3] a sequence $(x_j) \subset X$ is said to be *weak-polynomial* convergent to $x \in X$ if $P(x_j) \to P(x)$ for all $P \in \mathscr{P}(X)$; and the space X is defined to be a Λ-*space*, if whenever (x_j) is a sequence in X that is weak-polynomial convergent to 0, then $\|x_j\| \to 0$.

It is shown in [3] that l_p is a Λ-space for $1 \leq p < \infty$; it is also shown that $L_p(\mu)$ is a Λ-space for $2 \leq p < \infty$ and $L_1[0, 1]$ is not a Λ-space, and the question is posed as to whether $L_p(\mu)$ is a Λ-space for $1 < p < 2$. Our next result will provide an affirmative answer to this question.

First we recall that super-reflexive Banach spaces can be defined as those spaces that admit an equivalent uniformly convex norm. In particular, spaces $L_p(\mu)$ are super-reflexive for $1 < p < \infty$ and any measure μ (see, e.g., [8, Chapter 3]).

We shall use the following fact: If X is super-reflexive, then there exists

Received by the editors June 7, 1991 and, in revised form, September 23, 1991.
1991 *Mathematics Subject Classification*. Primary 46J15; Secondary 46B10.
Both authors were partially supported by DGICYT PB 87-1031.

©1993 American Mathematical Society
0002-9939/93 $1.00 + $.25 per page

some p $(1 < p < \infty)$ such that each bounded sequence (x_j) in X has a weakly p-convergent subsequence (x_{j_k}); that is, there exists $x \in X$ so that $\sum_{k=1}^{\infty} |x^*(x_{j_k} - x)|^p < +\infty$ for all $x^* \in X^*$. This is proved in [5, Proposition 3.1] using a suitable characterization of super-reflexivity given by James [12].

Theorem 1. *Every super-reflexive Banach space is a* Λ*-space.*

Proof. Let (x_j) be a sequence in a super-reflexive space X such that $P(x_j) \to P(0)$ for all $P \in \mathscr{P}(X)$, but $\|x_j\| \nrightarrow 0$. There exist some $\varepsilon > 0$ and a subsequence, which we still denote (x_j) such that $\|x_j\| \geq \varepsilon$; by the Bessaga-Pełczynski selection principle, (x_j) can be considered a basic sequence in X (see, e.g., [8]). Let Y denote the closed subspace of X spanned by (x_j), and let (x_j^*) be the corresponding sequence of functional coefficients associated to (x_j). Then (x_j^*) is a bounded sequence in Y^*. Each x_j^* can be extended to a functional $\hat{x}_j^* \in X^*$ with the same norm; therefore, $(\hat{x}_j^*)$ is a bounded sequence in X^*. X^*, however, is super-reflexive since X is (see, e.g., [16]) and then, as we remarked before, it follows from [5] that there exist some p $(1 < p < \infty)$, some $x^* \in X^*$, and a subsequence $(\hat{x}_{j_k}^*)$ such that

$$\sum_{k=1}^{\infty} |(\hat{x}_{j_k}^* - x^*)(x)|^p < +\infty \qquad \text{for all } x \in X.$$

Let $N \in \mathbb{N}$ with $N \geq p$. Then $\sum_{k=1}^{\infty} |(\hat{x}_{j_k}^* - x^*)(x)|^N < +\infty$ for all $x \in X$ and hence, by the Banach-Steinhaus type theorem for homogeneous polynomials (see, e.g., [6, 4.17]) the expression

$$P(x) = \sum_{k=1}^{\infty} ((\hat{x}_{j_k}^* - x^*)(x))^N, \qquad x \in X,$$

defines a continuous N-homogeneous polynomial on X. Now for each $y \in Y$ we have $y = \sum_{j=1}^{\infty} x_j^*(y) x_j$ and, therefore, $x_j^*(y) \to 0$. Since $(\hat{x}_{j_k}^* - x^*)(x) \to 0$ for each $x \in X$, it follows that $x^*(y) = 0$ for all $y \in Y$. In particular, we obtain that $P(x_{j_k}) = 1$ for all k, and this contradicts the hypothesis on the polynomial convergence of (x_j). □

Remark. In [5] a Banach space X is defined to be in the class W_p $(1 < p < \infty)$ when each bounded sequence in X admits a weakly p-convergent subsequence. The proof of Theorem 1 shows that if X^* is in the class W_p for some p $(1 < p < \infty)$ then X is a Λ-space. In particular, it follows from [5] that the dual Tsirelson space T and the space $(\bigoplus l_\infty^n)_{l_p}$ $(1 < p < \infty)$ are Λ-spaces, since the original Tsirelson space T^* is in W_p for all p $(1 < p < \infty)$ and $((\bigoplus l_\infty^n)_{l_p})^* = (\bigoplus l_1^n)_{l_{p^*}}$ is in W_p.

The authors thank Jesús F. Castillo for providing this remark (and other useful comments).

The notion of Λ-space was introduced in [3] in relation to the tightness of certain algebras of analytic functions on a (complex) Banach space. We recall that a uniform algebra A on a compact space K is said to be tight on K if, for all $g \in C(K)$, the Hankel-type operator $S_g: A \to C(K)/A$ defined by $S_g(f) = fg + A$ is weakly compact.

Now let Z be a complex dual Banach space with open unit ball B, and let $A(B)$ be the algebra generated by the weak*-continuous linear functionals on the closed unit ball $\overline{B}$ (regarded as functions on the weak*-compact set $\overline{B}$). It is proved in [3] that if $A(B)$ is tight on $\overline{B}$ then Z is reflexive. Therefore, we shall be mainly concerned with reflexive Banach spaces. It is also proved that if Z is an infinite-dimensional Λ-space with the metric approximation property, then $A(B)$ is not tight. We will obtain an extension of this last result for reflexive Z.

First we define a Banach space X to be a *κ-space* if there exists a weakly null sequence in X that is not weak-polynomial convergent to 0. In other words, X is a κ-space if and only if there exists a continuous polynomial P on X that is not weakly sequentially continuous; it is clear that P can be chosen to be m-homogeneous for some m.

The following results show that many classes of reflexive Banach spaces are κ-spaces.

Proposition 1. *Let X be a reflexive, infinite-dimensional Λ-space. Then X is a κ-space.*

Proof. Since X is an infinite-dimensional reflexive space, by the Josefson-Nissenzweig theorem (see, e.g., [9, Chapter 12]) there exists a weakly null sequence (x_j) in X that is not norm-convergent. Since X is Λ-space, (x_j) is not weak-polynomial convergent and, therefore, X is a κ-space. □

Proposition 2. *If X is a reflexive Banach space and a quotient Y of X is a κ-space, then X is a κ-space.*

Proof. Let $T: X \to Y$ be continuous, linear onto Y. Choose a sequence (y_j) in Y with $y_j \to 0$ weakly and $P \in \mathscr{P}(^mY)$ with $|P(y_j)| \geq \varepsilon$ for some $\varepsilon > 0$. We can apply the open mapping theorem to find a bounded sequence (x_j) in X with $(Tx_j) = (y_j)$. Since X is reflexive, there exist $x \in X$ and a subsequence (x_{j_k}) such that $x_{j_k} \to x$ weakly. Then $y_{j_k} = Tx_{j_k} \to Tx$ weakly and, therefore, $Tx = 0$. This shows that the polynomial $P \circ T \in \mathscr{P}(^mX)$ is not weakly sequentially continuous. □

Proposition 3. *Suppose that the Banach space X has a weakly null sequence (a_n) verifying that there exists a continuous linear operator $T: X \to l_p$ $(1 < p < \infty)$ such that (Ta_n) is the canonical basis of l_p. Then X is a κ-space.*

Proof. Choose $m \in \mathbb{N}$ with $m \geq p$. The series $\sum_{n=1}^{\infty} y_n^m = P(y)$, with $y = (y_n) \in l_p$, defines a continuous m-homogeneous polynomial on l_p. Then $Q = P \circ T \in \mathscr{P}(^mX)$ satisfies that $Q(a_n) = 1$ and therefore is not weakly sequentially continuous. □

Remark. Arguments of this kind have been used in [2, 1] to find a continuous polynomial of degree 2 that is not weakly sequentially continuous on the quasi-reflexive James space J and the dual Tsirelson space T, respectively. That is, J and T are κ-spaces. As remarked before, T is, in fact, a Λ-space.

We note that Proposition 3 applies whenever X is a Banach space of finite cotype with a weakly null, unconditional basis (see [14, 13]).

Proposition 4. *If a complemented subspace of a Banach space X is a κ-space, then X is a κ-space.*

Proof. Consider a continuous projection Π from X onto Y, where the subspace Y is a κ-space. We can choose a polynomial $P \in \mathscr{P}(Y)$ that is not weakly sequentially continuous. Then the polynomial $Q = P \circ \Pi \in \mathscr{P}(X)$ is not weakly sequentially continuous. □

This case covers a wide class of operator spaces defined on a κ-space. For example, the spaces $L(X)$ and $K(X)$, of bounded linear and compact linear operators on X, contain a complemented copy of X and, therefore, $L(X)$ and $K(X)$ are κ-spaces if X is.

An important class of nonreflexive Banach spaces are not κ-spaces. Recall that the Banach space X has the Dunford-Pettis property if whenever $(x_j) \subset X$ and $(\varphi_j) \subset X^*$ are weakly null sequences then $\varphi_j(x_j) \to 0$. Spaces that enjoy the Dunford-Pettis property are $C(K)$ and $L_1(\mu)$. Since the restriction of any polynomial on a Dunford-Pettis space to a weakly compact set is weakly continuous (see [15] or [3, 7.1]) it is clear that a space satisfying the Dunford-Pettis property is not a κ-space.

The proof of our next result follows the ideas appearing in [3, 9.4].

Theorem 2. *Let Z be a complex dual Banach space. Suppose that Z is a κ-space with the approximation property. Then $A(B)$ is not tight on $\overline{B}$.*

Proof. In view of [3, 9.1] we can assume that Z is reflexive. Since Z is a κ-space, for some m there exist a polynomial $P \in \mathscr{P}(^m Z)$ and a weakly null sequence $(z_j) \subset \overline{B}$ such that $P(z_j) \nrightarrow P(0) = 0$. Suppose that $A(B)$ is tight. Then by [3, 9.1] the $A(B)^{**}$-topology on $\frac{1}{2}\overline{B}$ (considered as a subset of $A(B)^*$) coincides with the weak topology (as a subset of Z). Then $F(\frac{1}{2}z_j) \to F(0)$ for all $F \in A(B)^{**}$. Now since Z is a dual space with the metric approximation property (see [7, p. 94]), we can apply [3, 3.1 and Theorem 4.4] to obtain that $A(B)^{**}|_B = \mathscr{H}^\infty(B)$; that is, each bounded holomorphic function on B is the restriction of some element of $A(B)^{**}$. Therefore $f(\frac{1}{2}z_j) \to f(0)$ for each $f \in \mathscr{H}^\infty(B)$; in particular, $2^{-m}P(z_j) = P(\frac{1}{2}z_j) \to 0$, which is a contradiction. □

Remark. Consider T^*, the original Tsirelson space; T^* is a reflexive space with an unconditional basis, which does not have any quotient isomorphic to l_p $(1 < p < \infty)$ and which is not a κ-space (since every continuous polynomial on T^* is weakly sequentially continuous; see [1]). Therefore T^* is a Banach space for which [3, 9.3 and 9.4] and our Theorem 2 cannot be applied. The tightness of $A(B)$, for T^*, will be studied below.

Now consider the space $E = T^* \times Z$, where Z is a reflexive κ-space with the approximation property. Since T^* is a closed subspace of E and T^* is not a Λ-space, neither is E. Nevertheless, E is a reflexive space with the approximation property and E is a κ-space (because of the same property of Z). So E provides an example of a Banach space satisfying our Theorem 2, which is not a Λ-space.

Proposition 5. *Let Z be the dual space of a complex separable Banach space. Suppose that there exists a point in the unit sphere $\overline{B}\backslash B$ of Z, which is not a complex extreme point. Then $A(B)$ is not tight on $\overline{B}$.*

Proof. Since $\overline{B}$ is weak-star metrizable, each point in $\overline{B}$ is a G_δ-set. In view of [10, II.12.1] a point in $\overline{B}\backslash B$ is a generalized peak point for $A(B)$ if and only if it is a peak point for $A(B)$. Now choose z in $\overline{B}\backslash B$ that is not a complex extreme point. Therefore, [11, Theorem 4] proves that z is not a peak point for $A(B)$ and, by [3, 9.5], $A(B)$ is not tight on $\overline{B}$. □

Proposition 5 provides the arguments showing that $A(B)$ is not tight for the unit ball B of the Tsirelson space T^*. In fact, it is enough to find z in $\overline{B}\backslash B$ such that z is not a complex extreme point:

Let (e_j) be the canonical unit basis of the dual Tsirelson space T and (e_j^*) the associated coefficient functionals of the original Tsirelson space T^*. By using the analytic description of the norm in T (see [4, I.1]), it is easy to check that $\|e_j + e_k\| = 1$ if $j \neq k$. Set $z = \frac{1}{2}(e_5^* + e_6^*)$ and $y = \frac{1}{2}(e_7^* + e_8^*)$. For each complex number λ with $|\lambda| \leq 1$, consider the vector $z + \lambda y$. Note that the natural projection of $z + \lambda y$ onto the first four coordinates is null, thus $\|2(z+\lambda y)\| = \|e_5^* + e_6^* + \lambda(e_y^* + e_8^*)\| \leq 2$ (see [1, Proposition 5; 4, p. 17]). Since $(z+\lambda y)(e_5 + e_6) = 1$, it is proved that $\|z + \lambda y\| = 1$ for each λ in the complex unit disc. It follows that z is not an extreme point and $A(B)$ is not tight.

Acknowledgments

The authors gratefully acknowledge several helpful conversations with Richard Aron and Miguel Lacruz while this paper was being prepared.

Note added in proof

Using Proposition 2.2 in [R. Aron, C. Hervés, M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. **52** (1983), 189–204], we can equivalently define the notion of κ-space as follows: A Banach space X is a κ-space if there exists a weakly Cauchy sequence in X that is not weak-polynomial convergent. Now, applying the Rosenthal-Dor l_1 theorem [9], we obtain the following sharper version of Proposition 2:

A Banach space not containing l_1, with a quotient being a κ-space, is a κ-space too.

The referee has kindly suggested this point of view.

References

1. R. Alencar, R. Aron, and S. Dineen, *A reflexive space of holomorphic functions in infinite many variables*, Proc. Amer. Math. Soc. **90** (1984), 407–411.
2. R. Aron and C. Hervés, *Weakly sequentially continuous analytic functions on a Banach space*, Functional Analysis, Holomorphy and Approximation Theory II (G. Zapata, ed.), North-Holland, Amsterdam, 1984, pp. 23–38.
3. T. Carne, B. Cole, and T. Gamelin, *A uniform algebra of analytic functions on a Banach space*, Trans. Amer. Math. Soc. **314** (1989), 639–659.
4. P. G. Casazza and T. J. Shura, *Tsirelson space*, Lecture Notes in Math., vol. 1363, Springer-Verlag, Berlin and New York, 1980.
5. J. F. Castillo and C. Sánchez, *Weakly-p-compact, p-Banach-Saks and super-reflexive Banach spaces*, preprint.
6. S. B. Chae, *Holomorphy and calculus in normed spaces*, Marcel Dekker, New York, 1985.
7. M. Day, *Normed linear spaces*, Springer-Verlag, Berlin and New York, 1973.
8. J. Diestel, *Geometry of Banach spaces*, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin and New York, 1975.
9. ——, *Sequences and series in Banach spaces*, Graduate Texts in Math., vol. 92, Springer-Verlag, Berlin and New York, 1984.
10. T. W. Gamelin, *Uniform algebras*, Chelsea, New York, 1984.
11. J. Globevnik, *On interpolation by analytic maps in infinite dimensions*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 243–254.

12. R. James, *Super-reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–420.

13. W. Johnson, *On finite dimensional subspaces of Banach spaces with local unconditional structure*, Studia Math. **51** (1974), 226–240.

14. B. Maurey and G. Pisier, *Séries de variables aléatories vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. **58** (1976), 45–90.

15. R. Ryan, *Dunford-Pettis properties*, Bull. Polon. Acad. Sci. **27** (1979), 373–379.

16. D. Van Dulst, *Reflexive and super-reflexive spaces*, Math. Centre Tracts, vol. 102, North-Holland, Amsterdam, 1982.

DEPARTMENTO DE ANALISIS MATEMATICO, UNIVERSIDAD COMPLUTENSE DE MADRID, 28040 MADRID, SPAIN

E-mail address, J. A. Jaramillo: JARAMIL@MAT.UCM.ES

E-mail address, A. Prieto: ANGELIN@EMDUCM11.BITNET