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THE ESSENTIAL SPECTRAL RADIUS OF DOMINATED POSITIVE OPERATORS

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ABSTRACT. Let E be an AL_p-space with $1 \leq p \leq \infty$. We prove that if a positive operator $S \in \mathscr{L}(E)$ satisfies the Doeblin conditions and $r(S) \leq 1$, then S is quasi-compact, i.e., $r_{\mathrm{ess}}(S) < 1$. We then deduce the following result about the monotonicity of the essential spectral radius: Let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$. If $r(S) \leq 1$ and $r_{\mathrm{ess}}(T) < 1$, then $r_{\mathrm{ess}}(S) < 1$.

0. INTRODUCTION

An interesting problem in the theory of positive operators in Banach lattices is to know what properties of $T \in \mathscr{L}(E)$, where E is a Banach lattice, are inherited by $S \in \mathscr{L}(E)$, if we know that $0 \leq S \leq T$. In 1979 Dodds and Fremlin [4] started the investigations in this direction by studying the behaviour of compactness under domination.

Dodds-Fremlin Theorem. *Let $S, T: E \to F$ be linear operators between two Banach lattices such that E' and F have order continuous norms. If $0 \leq S \leq T$ and T is compact, then S is compact.*

Compactness properties have also been considered in [1] and related spectral properties in [3, 5, 6]. In particular, it has been studied if $r_{\mathrm{ess}}(S) \leq r_{\mathrm{ess}}(T)$ when $0 \leq S \leq T$. The monotonicity of the essential spectral radius is known in only a few cases: when S is an AM-compact operator (see [2, 8]) and when E is an AM-space or an AL-space (see [6, 10]).

In [3] Caselles proved the following general result closely related to the monotonicity of the essential spectral radius:

Theorem 0.1. *Let E be a Banach lattice. Let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$. If $r(T) \leq 1$ and $r_{\mathrm{ess}}(T) < 1$, then $r_{\mathrm{ess}}(S) < 1$.*

Using the Doeblin condition, a similar result was stated in [5]:

Theorem 0.2. *Let $E = \mathscr{L}^p(\mu)$ with $1 \leq p \leq \infty$ and let $S, T \in \mathscr{L}(E)$ be such that $0 \leq S \leq T$. Assume that $r_{\mathrm{ess}}(T) < 1$ and that the means $S_n = \frac{1}{n} \sum_{i=0}^{n-1} S^i$ are uniformly bounded. Then $r_{\mathrm{ess}}(S) < 1$.*

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In this paper we prove Theorem 0.2 substituting the assumption $\sup\{\|S_n\|\} < \infty$ by the weaker one $r(S) \le 1$, as we conjectured in [6]. Thus our result generalizes Theorems 0.1 and 0.2.

1. Preliminaries

Throughout this paper, E will denote a complex Banach lattice, i.e., $E = E_{\mathbf{R}} \oplus \imath E_{\mathbf{R}}$, the complexification of a real Banach lattice $E_{\mathbf{R}}$. The absolute value in $E_{\mathbf{R}}$ is extended to E by means of the formula $|f| = \sup\{(\cos\theta)g + (\sin\theta)h : 0 \le \theta \le 2\pi\}$, where $f = g + \imath h$ with $g, h \in E_{\mathbf{R}}$ (see [9, Chapter II, 11] or [11, §91]) and the norm in E satisfies $\|z\| = \||z|\|$ for all $z \in E$. Given $x \in E^+$, by the complex order interval $[-x, x]$ we mean the subset $[-x, x] = \{z \in E : |z| \le x\}$.

Given $A \subseteq E$ we say that A is *order bounded* if A is included in some complex order interval. A is called *almost order bounded* if for every $\varepsilon > 0$ there exists $u > 0$ such that $A \subseteq [-u, u] + \varepsilon B_E$, B_E being the closed unit ball of E.

Let E be a complex Banach lattice. By $\mathscr{L}(E)$ we denote the space of all bounded linear operators from E into E. If $T \in \mathscr{L}(E_{\mathbf{R}})$ then T has a unique extension as a linear operator on E by defining

$$T(f + \imath g) = Tf + \imath Tg \quad \forall f, g \in E_{\mathbf{R}}.$$

Hence the set $\mathscr{L}(E_{\mathbf{R}})$ can be regarded as a real linear subspace of $\mathscr{L}(E)$. Any operator in this subspace is called real. For $T \in \mathscr{L}(E)$ there exist canonical real operators $\operatorname{Re} T$ and $\operatorname{Im} T$ such that $Tz = (\operatorname{Re} T)z + \imath(\operatorname{Im} T)z$ for all $z \in E$. An operator $T \in \mathscr{L}(E)$ is said to be positive ($T \ge 0$) if T is equal to $\operatorname{Re} T$ and $Tz \ge 0$ for $z \in E^+$.

If E is a complex Banach space and $T \in \mathscr{L}(E)$, we denote, as usual, the spectrum of T by $\sigma(T)$ and the spectral radius by $r(T)$. Recall that $T \in \mathscr{L}(E)$ is called upper (lower) semi-Fredholm if $T(E)$ is closed and its kernel (cokernel) $\operatorname{Ker} T(E/T(E))$ is finite dimensional. The set of all upper (lower) semi-Fredholm operators will be denoted by $\Phi_+(E)$ (respectively, $\Phi_-(E)$). $\Phi(E) = \Phi_+(E) \cap \Phi_-(E)$ is the set of the so-called *Fredholm operators.* The Fredholm domain of $T \in \mathscr{L}(E)$ is defined by

$$\Phi(T) = \{\lambda \in \mathbf{C} : (\lambda I - T) \in \Phi(E)\}.$$

The set $\mathbf{C} - \Phi(T)$ will be called the (Wolf) *essential spectrum* of T and will be denoted by $\sigma_{\text{ess}}(T)$. The essential spectrum of T is equal to the spectrum of the canonical image of T in the Calkin algebra $\mathscr{L}(E)/\mathscr{K}(E)$, where $\mathscr{K}(E)$ denotes the ideal of compact operators in E. The essential spectral radius is defined by

$$r_{\text{ess}}(T) = \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(T)\}.$$

We have that

$$r_{\text{ess}}(T) = \lim_n (\|T^n\|_{\text{ess}})^{1/n}, \quad \text{where } \|T\|_{\text{ess}} = \operatorname{dist}(T, \mathscr{K}(E)).$$

Finally, our general references concerning the theory of positive operators in Banach lattices are the monographs [1, 9, 11]. For the basic results on C_0-semigroups and further information, see [7].

2. The essential spectral radius and the Doeblin condition

The Doeblin conditions are the main tool that we use in order to study the monotonicity of the essential spectral radius. The following definitions are taken from [5].

Definition 2.1. Let T be a positive operator on the Banach lattice E.

(1) T is said to satisfy the Doeblin condition (which we note $T \in (D)$) if there exist $m \in \mathbf{N}$, $0 \leq \mu \in E'$, and a real number $\eta < 1$ such that

$$\|T^m x\| \leq \mu(x) + \eta\|x\| \quad \forall x \in E^+.$$

(2) T is said to satisfy the dual Doeblin condition (which we denote by $T \in (D')$) if there exist $m \in \mathbf{N}$, $0 \leq x \in E$, and a real number $\eta < 1$ such that

$$T^m B_E \subseteq [-x, x] + \eta B_E,$$

where B_E denotes the unit ball of E.

To prove our main result we need the following basic lemma. We omit its proof because it is a slight modification of Lemma V.5.3 in [9].

Lemma 2.2. *Let E be a Dedekind complete Banach lattice. Let $S \in \mathscr{L}(E)$ and let J be a band such that $S(J) \subseteq J$. Denote by $P_{J^\perp}$ the band projection onto $J^\perp$. If $r_{\text{ess}}(S/J) < 1$ and $r_{\text{ess}}(P_{J^\perp}S/J^\perp) < 1$, then $r_{\text{ess}}(S) < 1$.*

It is well known that if T is a positive linear operator on the Banach lattice E then the spectral radius $r(T)$ belongs to the spectrum $\sigma(T)$. However, in general, $r(T)$ is not an eigenvalue of T. In the next result, which is based on Proposition 2 in [5], we state the existence of a positive eigenvector when T satisfies the Doeblin conditions.

Theorem 2.3. *Let E be a Banach lattice and let S be a positive operator on E such that $S \in (D) \cap (D')$. If $r(S) = 1$ then there exists $a \in E^+$, $a \neq 0$ such that $Sa = a$.*

Proof. Since $S \geq 0$ and $r(S) = 1$, there exists a normalized sequence (x_n) of positive elements in E with $\lim_n \|(I - S)x_n\| = 0$. (Take, for example, $x_n = R(\lambda_n, S)x/\|R(\lambda_n, S)x\|$ where $x \geq 0$, $\lambda_n \to 1+$, $\|R(\lambda_n, S)x\| \to \infty$.)

Let $l^\infty(E)$ denote the Banach space of all bounded sequences (y_n) in E equipped with the supremum norm $\|(y_n)\|_\infty = \sup\{\|y_n\| : n \in \mathbf{N}\}$, and let $s(E)$ be the closed subspace of those sequences (y_n) in E such that their range $\{y_n : n \in \mathbf{N}\}$ is almost order bounded. We denote by $\widetilde{E}$ the quotient Banach space $l^\infty(E)/s(E)$. Since S is positive, S induces an operator $\widetilde{S}$ on $\widetilde{E}$ given by

$$\widetilde{S}((y_n) + s(E)) = (Sy_n) + s(E) \quad \forall (y_n) \in l^\infty(E)$$

(see [2]). As $S \in (D')$, we get $r(\widetilde{S}) < 1$. Now, since $((I - S)x_n) \in s(E)$, it follows that $(x_n) \in s(E)$.

Now let A be the closed ideal generated by E in E''. As $(x_n) \in s(E)$, given $\varepsilon > 0$, there exists $u \geq 0$ such that $\{x_n\} \subseteq [-u, u] + \varepsilon B_E$. It follows that

$$\overline{\{x_n\}}^{\sigma(E'', E')} \subseteq \overline{[-u, u] + \varepsilon B_E}^{\sigma(E'', E')} \subseteq [\![-u, u]\!] + \varepsilon B_{E''},$$

where $[\![-u, u]\!] = \{z \in E'' : |z| \leq u\}$. So we get $\overline{\{x_n\}}^{\sigma(E'', E')} \subseteq A$. Now, as $\overline{\{x_n\}}^{\sigma(E'', E')}$ is a $\sigma(E'', E')$-compact, there exist a subnet $\{x_{n_k}\}$ and $a \in A$ such that $\sigma(E'', E') - \lim_k x_{n_k} = a$, and so we obtain that $S''a = a$, $a \geq 0$.

Since $A \subseteq (E')^\sim_n$ (see [1, §5]), E is $|\sigma|(E'', E')$ dense in A (see [1, 11.16]). So there exists $\{y_\gamma\} \subseteq E$ such that $\{y_\gamma\} \subseteq [0, a]$ and $\sigma(E'', E')-\lim_\gamma |y_\gamma - a| = 0$. The Doeblin condition implies that, given $\varepsilon > 0$, there exist $\mu \in E'$, $n_0 \in \mathbf{N}$ such that

$$\|S^{n_0}y_\gamma - S^{n_0}y_\beta\| \leq \mu(|y_\gamma - y_\beta|) + \varepsilon\|y_\gamma - y_\beta\|.$$

Given $\phi \in B_{E'}$, we obtain

$$|\phi(S^{n_0}y_\gamma - S^{n_0}y_\beta)| \leq \mu(|y_\gamma - a|) + \mu(|y_\beta - a|) + \varepsilon\|y_\gamma - y_\beta\|.$$

It follows that there exists γ_0 such that if $\gamma, \beta \geq \gamma_0$ then

$$|((S^{n_0})'\phi)y_\gamma - \phi(S^{n_0}y_\beta)| \leq \varepsilon + 2\varepsilon \sup_\gamma \|y_\gamma\|.$$

Taking limits for γ, we get

$$|a((S^{n_0})'\phi) - \phi(S^{n_0}y_\beta)| \leq \varepsilon + 2\varepsilon \sup_\gamma \|y_\gamma\|,$$

and so

$$\|(S^{n_0})''a - S^{n_0}y_\beta\| \leq \varepsilon + 2\varepsilon \sup_\gamma \|y_\gamma\|.$$

Therefore we can write $(S^{n_0})''a = a \in E$ and, consequently, $\sigma(E, E') - \lim x_{n_k} = a$.

Since $Sa = a$, we only need to prove that $a \neq 0$. Suppose, on the contrary, that $a = 0$. Then the Doeblin condition implies that there exist $m \in \mathbf{N}$ and a real number $\eta < 1$ such that $\limsup_k \|S^m x_{n_k}\| \leq \eta < 1$. On the other hand, since $\lim_n \|(I - S)x_n\| = 0$, we get that $\lim_n \|(I - S^m)x_n\| = 0$, hence

$$\lim_k \|S^m x_{n_k}\| = \lim_k \|x_{n_k}\| = 1.$$

This contradiction concludes the proof.

Remark 2.4. (a) If E is a reflexive Banach lattice then Theorem 2.3 also holds when S satisfies only the dual Doeblin condition (see [5, Proposition 3]).

(b) If E is an AL-space (resp. an AM-space), then each operator satisfies the Doeblin condition (resp. the dual Doeblin condition).

Suppose $1 \leq p < \infty$. A Banach lattice E is called an *abstract $\mathscr{L}^p$-space* (or an *AL_p-space*) if

$$\|x + y\| = (\|x\|^p + \|y\|^p)^{1/p} \quad \forall x, y \in E^+ : x \wedge y = 0.$$

If $1 < p < \infty$ then any AL$_p$-space is reflexive. Of course, any concrete $\mathscr{L}^p(\mu)$-space is an abstract $\mathscr{L}^p$-space and it can be proved that the converse is also true (see [6, II, Example 23]). Obviously, each sublattice of an abstract $\mathscr{L}^p$-space is an abstract $\mathscr{L}^p$-space.

Our main result shows, roughly speaking, that a positive operator S in an abstract $\mathscr{L}^p$-space satisfying the Doeblin conditions is quasi-compact, i.e., $r_{\text{ess}}(S) < 1$. First we prove a simplified version making use of the so-called ultrapower $\widehat{E}$ of the Banach lattice E with respect to a free ultrafilter $\mathscr{U}$ on $\mathbf{N}$. For the construction of $\widehat{E}$, we refer to [9, V, 1.4].

Lemma 2.5. *Let E be an abstract $\mathscr{L}^p$-space with $1 \leq p < \infty$. Let $0 \leq S \in \mathscr{L}(E)$ such that S satisfies the dual Doeblin condition. If there exists a quasi-interior point x in E^+ such that $Sx = x$ and $r(S) = 1$, then $r_{\text{ess}}(S) < 1$.*

Proof. As $S \in (D')$, there exist $n_0 \in \mathbf{N}$, $y \in E^+$ such that

$$S^{n_0} B_E \subseteq [-y, y] + \tfrac{1}{2} B_E .$$

Since, by assumption, $y \in \overline{E_x}$, we obtain that there exists $m \in \mathbf{N}$ such that

$$S^{n_0} B_E \subseteq m[-x, x] + \tfrac{3}{4} B_E .$$

If we define now $z = m \sum_{n=0}^{\infty} (\frac{3}{4})^n x$, from the above expression we get

$$S^{n_0 n} B_E \subseteq [-z, z] + (\tfrac{3}{4})^n B_E .$$

Denoting by $\widehat{S}$ the canonical extension of S to the ultrapower $\widehat{E}$, we have

$$\widehat{S}^{n_0 n} B_{\widehat{E}} \subseteq [-\hat{z}, \hat{z}] + (\tfrac{3}{4})^n B_{\widehat{E}}$$

where $\hat{z}$ denotes the class of $(z, z, z, \ldots)$. Consequently, if $|\lambda| = 1$ then

$$\text{(1)} \qquad \operatorname{Ker}(\lambda - \widehat{S}) \cap B_{\widehat{E}} \subseteq [-\hat{z}, \hat{z}] .$$

We recall that $\widehat{E}_{\hat{z}}$, the principal ideal generated by $\hat{z}$, is an AM-space if we take $\{x' \in \widehat{E} : |x'| \leq \hat{z}\}$ as its unit ball. Note that in this case we have the following factorization:

$$\operatorname{Ker}(\lambda - \widehat{S}) \xrightarrow{i_1} \widehat{E}_{\hat{z}} \xrightarrow{i_2} \widehat{E}$$

where i_1, i_2 are the canonical inclusions. i_2 is by definition a continuous map and it follows from (1) that i_1 is a well-defined continuous operator.

(a) Suppose that E is an AL_p-space with $1 < p < \infty$. Since $\widehat{E}$ is an AL_p-space, the reflexivity of $\widehat{E}$ implies that i_1, i_2 are weakly compact operators. Since $\widehat{E}_{\hat{z}}$ is a Dunford-Pettis space, i_2 is a Dunford-Pettis map (see [1, Chapter 5, §19]). We conclude that $i_2 \circ i_1$ is a compact operator, whence $\operatorname{Ker}(\lambda - \widehat{S})$ is finite dimensional.

(b) Suppose that E is an AL-space. Then, as $\widehat{E}_{\hat{z}}$ is an AM-space and $\widehat{E}$ is an AL-space, $i_2 \circ i_1$ is an integral map (see [9, IV, Theorem 5.4]). Moreover, $\widehat{S} \in (D')$ implies that each bounded sequence in $\operatorname{Ker}(\lambda - \widehat{S})$ is almost order bounded. As $\widehat{E}$ has order continuous norm, each bounded sequence in $\operatorname{Ker}(\lambda - \widehat{S})$ is weakly compact. Therefore $\operatorname{Ker}(\lambda - \widehat{S})$ is reflexive. It follows that $i_2 \circ i_1$ is compact (see [9, IV, Theorem 5.4, Corollary 1 and Theorem 5.5, Corollary 2]). Therefore $\operatorname{Ker}(\lambda - \widehat{S})$ is finite dimensional.

Theorem 3.4 in [3] implies now that $\lambda \notin \sigma_{\text{ess}}(S)$ whence $r_{\text{ess}}(S) < 1$.

We now establish our main result:

Theorem 2.6. *Let E be an AL_p-space with $1 \leq p < \infty$. Let S be a positive operator on E such that $r(S) \leq 1$. If S satisfies the dual Doeblin condition then $r_{\text{ess}}(S) < 1$.*

Proof. Since $r_{\text{ess}}(S) \leq r(S)$, we can assume that $r(S) = 1$. Then Remark 2.4 shows that there exists a positive element $a_1 \neq 0$ in E such that $Sa_1 = a_1$. If E_{a_1} denotes the principal ideal generated by a_1 then $\overline{E}_{a_1}$ is a closed ideal (in fact, $\overline{E}_{a_1}$ is a band because E has order continuous norm) with quasi-interior points. It is easy to see that $S/\overline{E}_{a_1} \in (D')$, and then it follows from Lemma 2.5 that $r_{\text{ess}}(S/\overline{E}_{a_1}) < 1$. Now let us consider the operator $(P_{(E_{a_1})^\perp}S)/(E_{a_1})^\perp \in \mathscr{L}((E_{a_1})^\perp)$, where $P_{(E_{a_1})^\perp}$ denotes the band projection onto $E_{a_1}^\perp$. If $r((P_{(E_{a_1})^\perp}S)/(E_{a_1})^\perp) < 1$ then Lemma 2.2 implies $r_{\text{ess}}(S) < 1$, which concludes the proof.

Assume on the contrary that

$$r((P_{(E_{a_1})^\perp}S)/E_{a_1})^\perp) = 1.$$

Since $(P_{(E_{a_1})^\perp}S)/(E_{a_1})^\perp \in (D')$, we get from Remark 2.4 that there exists $a_2 > 0$ such that $(P_{(E_{a_1})^\perp}S)/(E_{a_1})^\perp a_2 = a_2$. Denote by $\overline{E}_{a_2}$ the closed ideal (band) generated by a_2. We can write

$$P_{\overline{E}_{a_2}}Sa_2 = a_2, \qquad S(\overline{E}_{a_1} \oplus \overline{E}_{a_2}) \subseteq E_{a_1} \oplus E_{a_2}.$$

Lemma 2.5 implies that $r_{\text{ess}}(P_{\overline{E}_{a_2}}S/\overline{E}_{a_2}) < 1$ and, as $\overline{E}_{a_2}$ is the band orthogonal to $\overline{E}_{a_1}$ in the Banach lattice $\overline{E}_{a_1} \oplus \overline{E}_{a_2}$, it follows from Lemma 2.2 that $r_{\text{ess}}(S/\overline{E}_{a_1} \oplus \overline{E}_{a_2}) < 1$. As a consequence of the previous process, one of the following assertions must be true:

(a) $r_{\text{ess}}(S) < 1$; or

(b) there exists a sequence of positive disjoint elements $\{a_n\}$ in E such that

$$S(\overline{E}_{a_1} \oplus \overline{E}_{a_2} \oplus \cdots \oplus \overline{E}_{a_1}) \subseteq \overline{E}_{a_1} \oplus \overline{E}_{a_2} \oplus \cdots \oplus \overline{E}_{a_n},$$
$$Sa_n \geq a_n, \qquad \|a_n\| = 1, \qquad a_n \geq 0 \quad \forall n \in \mathbf{N}, \qquad a_n \perp a_m \quad \text{if } n \neq m.$$

Suppose that (b) holds. In this case, we take the class $\tilde{a}$ of the sequence $\{a_n\}$ in the Banach lattice $\widetilde{E} = l_\infty(E)/s(E)$. Since $0 \leq \tilde{a} \leq \widetilde{S}^n\tilde{a} \forall n \in \mathbf{N}$ and $r(\widetilde{S}) < 1$ (note that $S \in (D')$), $\tilde{a}$ must be equal to 0, whence $\{a_n\} \in s(E)$. Therefore, given $0 < \varepsilon < 1$, there exist $y_n, z_n, x \in E$ such that

$$a_n = y_n + z_n, \qquad 0 \leq y_n \leq x \ \forall n \in \mathbf{N}, \ \sup_n \|z_n\| \leq \varepsilon.$$

As $\{y_n\}$ is an order bounded disjoint sequence, we get $\lim_n \|y_n\| = 0$ (see [1, Theorem 12.13]). So $\limsup_n \|a_n\| \leq \varepsilon < 1$, which is a contradiction.

Corollary 2.7. *Let E be an AL_p-space with $1 < p \leq \infty$ or an AM-space. Let S be a positive operator on E such that $r(S) \leq 1$. If S satisfies the Doeblin condition then $r_{\text{ess}}(S) < 1$.*

Proof. Since S' satisfies conditions of Theorem 2.6, it follows that $r_{\text{ess}}(S') = r_{\text{ess}}(S) < 1$.

Note that every quasi-compact operator T satisfies the Doeblin conditions and also that when $0 \leq S \leq T$, $S \in (D)$ (resp. (D')) if $T \in (D)$ (resp. (D')). The following corollary is then immediate.

Corollary 2.8. *Let E be an AL_p-space with $1 \leq p \leq \infty$ or an AM-space, and let $S, T \in \mathscr{L}(E)$ be positive operators such that $0 \leq S \leq T$. If $r_{\text{ess}}(T) < r(S)$ then $r_{\text{ess}}(S) < r(S)$.*

Remark 2.9. It is unknown if Corollary 2.8 holds for arbitrary Banach lattices.

3. Quasi-compactness of dominated positive C_0-semigroups

In this section we are going to apply the results of the above section to positive C_0-semigroups. We recall first some standard definitions and notations (see [6]).

Let $\{T(t) : t \geq 0\}$ be a C_0-semigroup of bounded linear operators on the Banach space E with generator A. The growth bound $\omega(A)$ of the C_0-semigroup $\{T(t)\}$ is defined by

$$\begin{aligned}\omega(A) &= \inf\{\omega \in \mathbf{R} : \exists M \in \mathbf{R}^+ \text{ such that } \|T(t)\| \leq Me^{\omega t} \forall t \geq 0\} \\ &= \lim_t \frac{1}{t} \log \|T(t)\|\end{aligned}$$

and the essential bound, $\omega_{\text{ess}}(A)$, by

$$\omega_{\text{ess}}(A) = \lim_t \frac{1}{t} \log \|T(t)\|_{\text{ess}}.$$

It is known that

$$r(T(t)) = e^{t\omega(A)} \quad \text{and} \quad r_{\text{ess}}(T(t)) = e^{t\omega_{\text{ess}}(A)}.$$

These two bounds are related via

$$\omega(A) = \max(\omega_{\text{ess}}(A), s(A))$$

where $s(A)$ is the spectral bound of the generator A, that is,

$$s(A) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

When dealing with C_0-semigroups, Theorem 2.6 can be rewritten as

Theorem 3.1. *Let E be an AL_p-space with $1 \leq p < \infty$. Let $\{T(t)\}$ be a positive C_0-semigroup on E with generator A. Assume that there exist $t_0 \geq 0$, $u \in E^+$, and $\eta < 1$ such that*

$$T(t_0)B_E \subseteq [-u, u] + \eta B_E.$$

Then, if $\omega(A) \leq 0$, $\{T(t)\}$ is quasi-compact, i.e., $\omega_{\text{ess}}(A) < 0$.

Analogously, it can also be proved:

Theorem 3.2. *Let E be an AL_p-space with $1 < p \leq \infty$ or an AM-space. Let $\{T(t)\}$ be a positive C_0-semigroup on E with generator A. Assume that there exist $t_0 \geq 0$, $0 \leq \mu \in E'$, and $\eta < 1$ such that*

$$\|T(t_0)f\| \leq \mu(f) + \eta\|f\| \quad \forall f \in E^+.$$

Then, if $\omega(A) \leq 0$, $T(t)$ is quasi-compact, i.e., $\omega_{\text{ess}}(A) < 0$.

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