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REFINEMENTS OF THE DENSITY AND $\mathscr{I}$-DENSITY TOPOLOGIES

KRZYSZTOF CIESIELSKI AND LEE LARSON

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ABSTRACT. Given an arbitrary ideal $\mathscr{I}$ on the real numbers, two topologies are defined that are both finer than the ordinary topology. There are nonmeasurable, non-Baire sets that are open in all of these topologies, independent of $\mathscr{I}$. This shows why the restriction to Baire sets is necessary in the usual definition of the $\mathscr{I}$-density topology. It appears to be difficult to find such restrictions in the case of an arbitrary ideal.

In studying real functions it is often helpful to endow the real numbers $\mathbb{R}$ with a topology finer than the natural one, denoted by $\mathscr{T}_{\mathscr{O}}$, arising from its order. The most common examples of such topologies are the density topology $\mathscr{T}_{\mathscr{N}}$ and its category analog, the $\mathscr{I}$-density topology $\mathscr{T}_{\mathscr{I}}$.

To recall the definitions of these topologies we need the following notions. A point $x \in \mathbb{R}$ is said to be a *density point* of $A \subset \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{\mathrm{m}(A \cap (x - 1/n, x + 1/n))}{2/n} = 1, \tag{1}$$

or, equivalently,

$$\lim_{h\to 0^+} \frac{\mathrm{m}(A \cap (x - h, x + h))}{2h} = 1, \tag{2}$$

where $\mathrm{m}(A)$ denotes the inner Lebesgue measure of A.

We let $\Phi_{\mathscr{N}}(A)$ be the set of all density points of $A \subset \mathbb{R}$. The density topology is defined by $\mathscr{T}_{\mathscr{N}} = \{A \subset \mathbb{R} : A \subset \Phi_{\mathscr{N}}(A)\}$. It is a well-known theorem that $\mathscr{T}_{\mathscr{N}} \subset \mathscr{L}$, where $\mathscr{L}$ stands for the family of all Lebesgue measurable sets.

To motivate the definition of the $\mathscr{I}$-density topology $\mathscr{T}_{\mathscr{I}}$, the following reformulation of the definition of a density point is usually used [6, 8].

For a measurable set A the following conditions are equivalent:

(A) 0 is a density point of A;

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(B) $\lim_{n\to\infty} \mathrm{m}(A \cap (-1/n, 1/n))/(2/n) = 1$;
(C) $\lim_{n\to\infty} \mathrm{m}(nA \cap (-1, 1)) = 2$;
(D) $\chi_{nA\cap(-1,1)}$ converges to $\chi_{(-1,1)}$ in measure; and
(E) for every increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{n_{m_p}A\cap(-1,1)} = \chi_{(-1,1)} \quad \text{a.e.}$$

The equivalence of (D) and (E) follows from a well-known theorem of Riesz concerning convergence in measure (or stochastic convergence) [1, Theorem 2.11.6].

Using the translation invariance of Lebesgue measure, the same sequence of equivalences can be rewritten for any density point of A. The significance of the equivalence of (E) and (A) is that it clearly shows the measure function itself is not vital to the definition of a density point. What are needed is the σ-algebra $\mathscr{L}$ of measurable sets and the ideal $\mathscr{N}$ of measure zero sets.

So, let us make an attempt to define a $\mathscr{J}$-density topology for an arbitrary ideal $\mathscr{J}$ of subsets of $\mathbb{R}$. We start with the following definitions.

A proposition is said to be *true $\mathscr{J}$-a.e.* if the set of points at which it does not hold is a member of $\mathscr{J}$; more formally, a Boolean function P is true $\mathscr{J}$-a.e. if, and only if, $\{x : \neg P(x)\} \in \mathscr{J}$.

If f_n is a sequence of real-valued functions, we say that f_n *converges* $(\mathscr{J})$ to a function f if for every subsequence f_{n_p} of f_n, there exists a further subsequence $f_{n_{p_q}}$ such that $f_{n_{p_q}}$ converges pointwise to f, $\mathscr{J}$-a.e.

Finally, a point a is a *$\mathscr{J}$-density point* of a set $S \subset \mathbb{R}$ if $\chi_{n(S-a)\cap(-1,1)}$ converges $(\mathscr{J})$ to $\chi_{(-1,1)}$. Let $\Phi_{\mathscr{J}}(S)$ denote the set of all $\mathscr{J}$-density points of the set S.

In this way we have defined a $\mathscr{J}$-density that is analogous to ordinary Lebesgue density. In fact, when the ideal $\mathscr{J}$ is taken to be the ideal of Lebesgue null sets $\mathscr{N}$ and the set S is measurable, then the equivalence of (A) with (E) shows that the $\mathscr{J}$-density points are precisely the points of ordinary Lebesgue density. To continue the analogy, let $\mathscr{T}'_{\mathscr{J}} = \{S \subset \mathbb{R} : S \subset \Phi_{\mathscr{J}}(S)\}$. It is quite easy to see that the family $\mathscr{T}'_{\mathscr{J}}$ is a topology on $\mathbb{R}$ and that in the case $\mathscr{J} = \mathscr{N}$, it contains the ordinary density topology; i.e., that $\mathscr{T}_{\mathscr{N}} \subset \mathscr{T}'_{\mathscr{N}}$.

At this point, one might suspect that the analogy will continue and that the properties of $\mathscr{T}'_{\mathscr{J}}$ could be developed very naturally along the same lines as the Lebesgue density topology. However, in the usual definition of the $\mathscr{J}$-density topology, $\mathscr{J}$ being the ideal of first category sets, it is additionally assumed that the sets in $\mathscr{T}_{\mathscr{J}}$ have the Baire property; i.e., the $\mathscr{J}$-density topology is defined by

$$\mathscr{T}_{\mathscr{J}} = \mathscr{T}'_{\mathscr{J}} \cap \mathscr{B} = \{S \in \mathscr{B} : S \subset \Phi_{\mathscr{J}}(S)\},$$

where $\mathscr{B}$ stands for the family of all subsets of $\mathbb{R}$ with the Baire property.

It is natural to ask whether this additional assumption in the definition is essential. The following theorem shows this is really the case.

Theorem 1. *There exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property, such that $\lim_{n\to\infty} \chi_{n(A-a)\cap(-1,1)} = \chi_{(-1,1)}$ for every $a \in A$.*

Proof. Let B be a Hamel basis that is also a Bernstein set; i.e., a linear basis of $\mathbb{R}$ over $\mathbb{Q}$, such that B intersects every nonempty perfect set.[1] For $x \in \mathbb{R}$ let $\rho'(x) = \sum_{i=1}^{n} |\alpha_i|$ where $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$ is a representation of x in the base B. Define $A = \{x \in \mathbb{R}\colon \rho'(x) < s\}$, where $s \in (1/2, 1)$. Obviously $\frac{1}{2}B \subset A$, so that A intersects every nonempty perfect set. Also, $B \subset \mathbb{R} \setminus A$, and thus the complement of A also intersects every nonempty perfect set. This proves that A neither is Lebesgue measurable nor has the Baire property.

Let $a \in A$. We will prove that

$$\lim_{n\to\infty} \chi_{n(A-a)\cap(-1,1)} = \chi_{(-1,1)}$$

everywhere.

Let $x \in (-1, 1)$. There is a natural number n_0 such that $\frac{1}{n}x + a \in A$, for all $n \geq n_0$, $n \in \mathbb{N}$. In fact, for $x = \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_k b_k$ and $a = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_k b_k$ it suffices to choose n_0 such that

$$\sum_{i=1}^{k} |\alpha_i| + \frac{1}{n_0} \sum_{i=1}^{k} |\beta_i| < s.$$

Then, for every $n \geq n_0$,

$$x = n((\tfrac{1}{n}x + a) - a) \in n(A - a).$$

Let $\mathcal{O} = \{\varnothing\}$. As an immediate corollary we obtain

Corollary 2. *$\mathcal{T}'_{\mathcal{O}} \not\subset \mathcal{T}_{\mathcal{N}} \cap \mathcal{T}_{\mathcal{J}}$. In particular, $\mathcal{T}'_{\mathcal{N}} \not\subset \mathcal{T}_{\mathcal{N}}$ and $\mathcal{T}'_{\mathcal{J}} \not\subset \mathcal{T}_{\mathcal{J}}$.*

It is also easy to see that if s is chosen to be irrational in the proof of the previous theorem then the complement A^c of A satisfies a condition similar to (A). Thus, although the topologies $\mathcal{T}_{\mathcal{N}}$ and $\mathcal{T}_{\mathcal{J}}$ are connected [2, 8], this is not the case with $\mathcal{T}'_{\mathcal{N}}$ and $\mathcal{T}'_{\mathcal{J}}$, as stated below.

Corollary 3. *Let $\mathcal{J}$ be an arbitrary ideal. Then there exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property, such that $A, A^c \in \mathcal{T}'_{\mathcal{O}} \subset \mathcal{T}'_{\mathcal{J}}$. In particular, $\mathcal{T}'_{\mathcal{J}}$ is disconnected.*

The set A in this corollary is a "universal" clopen set in all the topologies $\mathcal{T}'_{\mathcal{J}}$. In particular, $\mathcal{T}_{\mathcal{N}} \neq \mathcal{T}'_{\mathcal{N}}$. So the logical question is, why is it avoided in the ordinary density topology? A careful reading of the equivalences (A) and (E) shows that those equivalences are only valid when the set is assumed to be Lebesgue measurable. Thus, the ordinary density topology is all sets S such that $S \subset \Phi_{\mathcal{N}}(S)$ *and* S is measurable, which excludes the set defined in Theorem 1. As we mentioned before, this restriction to measurable sets is a consequence of the normal definition of the density topology, contained in (1), but it is lost with the more general approach, based on ideals.

The above arguments also justify the definition of the $\mathcal{J}$-density topology as $\mathcal{T}_{\mathcal{J}} = \mathcal{T}'_{\mathcal{J}} \cap \mathcal{B}$. In addition, by letting $f = \chi_A$, where A is from Corollary 3, the following corollary is easily seen.

[1] To construct such a basis $B = \{b_\zeta : \zeta < c\}$ it is enough to define it by transfinite induction on ζ. We can simply choose b_ζ from $P_\zeta \setminus \mathbb{Q}(\{b_\xi : \xi < \zeta\})$, where $\{P_\zeta : \zeta < c\}$ is a fixed enumeration of the nonempty perfect subsets of $\mathbb{R}$ and $\mathbb{Q}(\{b_\xi : \xi < \zeta\})$ stands for a subfield of $\mathbb{R}$ generated by $\{b_\xi : \xi < \zeta\}$. Compare also [5].

Corollary 4. *If $\mathcal{J} \in \{\mathcal{I}, \mathcal{N}\}$, then family of all continuous $f\colon (\mathbb{R}, \mathcal{T}'_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$ is strictly larger than both $f\colon (\mathbb{R}, \mathcal{T}_{\mathcal{J}}) \to (\mathbb{R}, \mathcal{T}_{\mathcal{O}})$.*

Perhaps in the equivalences (A) through (E) we should replace the definition (1) by (2) in condition (B)? If we do, we obtain the following conditions that are equivalent for every measurable set A.

(A) 0 is a density point of A;

(B′) for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to ∞,

$$\lim_{n\to\infty} \frac{\mathrm{m}(A \cap (-1/t_n, 1/t_n))}{2/t_n} = 1;$$

(C′) for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to ∞, $\lim_{n\to\infty} \mathrm{m}(t_n A \cap (-1, 1)) = 2$;

(D′) for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to ∞, $\chi_{t_n A\cap(-1,1)}$ converges to $\chi_{(-1,1)}$ in measure; and

(E′) for every sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers diverging to ∞ there exists a subsequence $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that

$$\lim_{p\to\infty} \chi_{t_{n_p} A\cap(-1,1)} = \chi_{(-1,1)} \quad \text{a.e.}$$

To follow the path already familiar from the previous definitions, let $\{t_n\}$ be any sequence of positive numbers diverging to ∞ and for any $S \subset \mathbb{R}$, define $\Psi_{\mathcal{J}}(S, \{t_n\})$ to be the set of all $a \in S$ such that $\chi_{t_n(S-a)\cap(-1,1)}$ converges ($\mathcal{J}$) to $\chi_{(-1,1)}$. Then set

$$\Psi_{\mathcal{J}}(S) = \bigcap_{\{t_n\}} \Psi_{\mathcal{J}}(S, \{t_n\}),$$

where the intersection is over all sequences $\{t_n\}$ of positive numbers diverging to ∞. It is not hard to show that $\mathcal{T}''_{\mathcal{J}} = \{S \subset \mathbb{R} : S \subset \Psi_{\mathcal{J}}(S)\} \subset \mathcal{T}'_{\mathcal{J}}$ is a topology on $\mathbb{R}$. This is termed an *abstract $\mathcal{J}$-density topology.*

It follows, by the equivalence of (E), (E′), and (A), that $\mathcal{T}_{\mathcal{N}} = \mathcal{T}'_{\mathcal{N}} \cap \mathcal{L} = \mathcal{T}''_{\mathcal{N}} \cap \mathcal{L}$. It is also known that $\mathcal{T}_{\mathcal{J}} = \mathcal{T}'_{\mathcal{J}} \cap \mathcal{B} = \mathcal{T}''_{\mathcal{J}} \cap \mathcal{B}$ [7, 8]. But we may hope that at least $\mathcal{T}''_{\mathcal{N}} = \mathcal{T}_{\mathcal{N}}$ or $\mathcal{T}''_{\mathcal{J}} = \mathcal{T}_{\mathcal{J}}$. As the following theorem shows, however this is not the case.

Theorem 5. *There exists a nonmeasurable set $A \subset \mathbb{R}$, which does not have the Baire property, such that $A \subset \Psi_{\mathcal{I}_c}(A)$, where $\mathcal{I}_c$ is the ideal of all sets of cardinality less than the continuum c.*

Proof. Let $B = \{b_{\zeta+1} : \zeta < c\}$ be a transcendental base of $\mathbb{R}$ over $\mathbb{Q}$ such that B intersects every perfect set.[2] For $\xi < c$ let K_ξ be the algebraic closure of $\{b_\zeta : \zeta < \xi\}$ in $\mathbb{R}$, and let B^ξ be a linear base of $K_{\xi+1}$ over K_ξ containing 1 and b_ξ. For $a \in K^{\xi+1} \setminus K^\xi$, let $\rho''(a) = |\alpha_0|$, where $a = \alpha_0 b_\xi + \alpha_1 b_1 + \cdots + \alpha_k b_k$ is a representation of a in the base B^ξ.

Define $A = \{a \in \mathbb{R} : \rho''(a) < 1\}$. As in Theorem 1, $\frac{1}{2}B \subset A$ and $B \subset \mathbb{R} \setminus A$, so that A neither is measurable nor has the Baire property. Note that if $x \in K^\xi$ and $y \notin K^\xi$ then $\rho''(x + y) = \rho''(y)$. Let $a \in A$ and let $\{t_n\}_{n\in\mathbb{N}}$ be an increasing sequence diverging to infinity. Let $\xi < c$ be such that $a, t_n \in K^\xi$

[2]For the definition of transcendental base see, e.g., [4]. The additional requirements are obtained in the same way as described in the footnote for Theorem 1.

for every $n \in \mathbb{N}$. We have $\operatorname{card}(K^\xi) < c$. It suffices to show that for every $x \in (-1, 1) \setminus K^\xi$

$$\lim_{n\to\infty} \chi_{t_n(A-a)\cap(-1,1)}(x) = \chi_{(-1,1)}(x) = 1.$$

It is enough to show that $x \in t_n(A-a)$ for all but finitely many n. This is equivalent to the fact that $\frac{x}{t_n} + a \in A$. But, $1/t_n \in K_\xi$, so that $x/t_n \notin K_\xi$, and $a \in K_\xi$. Therefore

$$\rho''\left(\frac{x}{t_n} + a\right) = \rho''\left(\frac{x}{t_n}\right) = \frac{\rho''(x)}{|t_n|}.$$

If we choose $n \in \mathbb{N}$ such that $1/|t_n| < 1$, then $x \in t_n(A-a)$. □

In particular, the previous theorem implies

Corollary 6. *If the Continuum Hypothesis, or Martin's Axiom, holds, then there is a nonmeasurable set A without the Baire property such that $A \subset \Psi_{\mathscr{I}}(A) \cap \Psi_{\mathscr{N}}(A)$; i.e., $A \in \mathscr{T}''_{\mathscr{N}} \cap \mathscr{T}''_{\mathscr{I}} \setminus (\mathscr{L} \cup \mathscr{B})$.*

Proof. If the Continuum Hypothesis holds, then

$$\mathscr{I}_c = \{B : \operatorname{card}(B) \le \aleph_0\} \subset \mathscr{I} \cap \mathscr{N}.$$

Martin's Axiom also implies $\mathscr{I}_c \subset \mathscr{I} \cap \mathscr{N}$. (See [3].) □

The previous examples show that the topology $\mathscr{T}''_{\mathscr{I}}$ does not behave as nicely as the ordinary density topology even when $\mathscr{I} = \mathscr{N}$. Thus, to obtain a good analogy, we must refine the definition somewhat. However, the topologies $\mathscr{T}''_{\mathscr{I}}$ perhaps deserve closer scrutiny. For example, the following open problems seem to be interesting.

Problem 1. Are the topologies $\mathscr{T}'_{\mathscr{I}}$, $\mathscr{T}'_{\mathscr{N}}$, $\mathscr{T}''_{\mathscr{I}}$, and $\mathscr{T}''_{\mathscr{N}}$ on $\mathbb{R}$ regular? completely regular? normal?

Problem 2. Are the topologies $\mathscr{T}''_{\mathscr{I}}$ and $\mathscr{T}''_{\mathscr{N}}$ disconnected?

Problem 3. Can we prove, without additional set theoretical assumptions, that $\mathscr{T}''_{\mathscr{I}} \neq \mathscr{T}_{\mathscr{I}}$ and $\mathscr{T}''_{\mathscr{N}} \neq \mathscr{T}_{\mathscr{N}}$?

The above discussion shows clearly that in order to define a "reasonable" $\mathscr{J}$-density topology $\mathscr{T}_{\mathscr{J}}$ we should define it as $\mathscr{T}_{\mathscr{J}} = \mathscr{T}''_{\mathscr{J}} \cap \mathscr{F}_{\mathscr{J}}$ for some family $\mathscr{F}_{\mathscr{J}}$ of subsets of $\mathbb{R}$. However, there is an evident problem with this definition. How do we correctly choose the family $\mathscr{F}_{\mathscr{J}}$ for the given ideal $\mathscr{J}$? Our choice should at least guarantee that the family

$$\mathscr{T}_{\mathscr{J}} = \mathscr{T}''_{\mathscr{J}} \cap \mathscr{F}_{\mathscr{J}} \quad \text{is a topology on } \mathbb{R}. \tag{3}$$

It would be also very desirable to have the equation

$$\mathscr{T}_{\mathscr{J}} = \mathscr{T}'_{\mathrm{J}} \cap \mathscr{F}_{\mathscr{J}}. \tag{4}$$

In the remaining part of this note we will argue that for a general ideal $\mathscr{J}$, finding the natural and nontrivial family $\mathscr{F}_{\mathscr{J}}$ satisfying conditions (3) and (4) could be very difficult, if not impossible.

The easiest example of these difficulties comes from the ordinary topology $\mathscr{T}_{\mathscr{O}}$. In particular, it is clear from the next proposition that we cannot even allow the family $\mathscr{F}_{\mathscr{O}}$ to contain all complements of converging sequences and simultaneously have condition (4).

Proposition 7. *There is a decreasing sequence S converging to 0 such that $S^c \in \mathscr{T}'_{\mathscr{O}}$. Moreover, $\mathscr{T}_{\mathscr{O}} = \mathscr{T}''_{\mathscr{O}}$. In particular, $\mathscr{T}'_{\mathscr{O}} \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta \not\subset \mathscr{T}_{\mathscr{O}} = \mathscr{T}''_{\mathscr{O}}$.*

Proof. The inclusion $\mathscr{T}_{\mathscr{O}} \subset \mathscr{T}''_{\mathscr{O}}$ is obvious. To see the reverse containment let us take $A \notin \mathscr{T}_{\mathscr{O}}$. Then, there exists a point $a \in A$ and a monotone sequence $\{p_n\} \subset A^c$ converging to a. Translating the set A, if necessary, we may assume that $a = 0$. Then, it is easy to see that $0 \notin \Psi_{\mathscr{O}}(A, \{|p_n^{-1}|/2\})$; i.e., $A \notin \mathscr{T}''_{\mathscr{O}}$.

For the second part it is enough to choose a decreasing sequence S converging to 0 such that the set S is linearly independent over $\mathbb{Q}$. Then $\chi_{nS^c \cap (-1,1)}$ converges everywhere to $\chi_{(-1,1)}$, because for every point $x \in (-1, 1)$ there is at most one n such that $x \in nS^c$. Hence, $S^c \in \mathscr{T}'_{\mathscr{O}}$. □

To see how bad things can be, consider the ideal $\mathscr{I}_\omega$ of countable subsets of $\mathbb{R}$. Then we have

Theorem 8. *There exist nonempty perfect sets P and C such that $\{0\} \cup P^c \in \mathscr{T}'_{\mathscr{I}_\omega} \setminus \mathscr{T}''_{\mathscr{I}_\omega}$ and $Z^c \in \mathscr{T}''_{\mathscr{I}_\omega}$ for every set $Z \subset C$. In particular,*

$$\mathscr{T}'_{\mathscr{I}_\omega} \cap \mathbf{F}_\sigma \cap \mathbf{G}_\delta \not\subset \mathscr{T}''_{\mathscr{I}_\omega} \quad \text{and} \quad \mathscr{T}''_{\mathscr{I}_\omega} \not\subset \mathscr{B}.$$

Proof. We start the proof with the construction of the set C. Define by induction on n a decreasing sequence of closed sets $\{C_n\}_{n\in\mathbb{N}}$ such that each C_n is formed from 2^n pairwise disjoint closed subintervals of $[0, 1]$ of equal length. Put $C_0 = [0, 1]$. To form C_{n+1} from C_n we remove from every component interval $[a, b]$ of C_n the open interval (c, d) with the same center and such that $\frac{d-c}{b-a} = \frac{(n+2)!-1}{(n+2)!}$. Put $C = \bigcap_{n\in\mathbb{N}} C_n$.

It is not difficult to check that for every $x \in C$ there an interval set $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n] \cup \bigcup_{n\in\mathbb{N}}[c_n, d_m] \supset C$ such that $a_n < b_n < a_{n+1} < x < d_{n+1} < c_n < d_n$ for every $n \in \mathbb{N}$,

$$\lim_{n\to\infty} (b_n - a_n)/(x - b_n) = \lim_{n\to\infty} (d_n - c_n)/(c_n - x) = 0$$

and

$$\lim_{n\to\infty} (x - a_{n+1})/(x - b_n) = \lim_{n\to\infty} (d_{n+1} - x)/(c_n - x) = 0.$$

But, it is a consequence of the proof of [8, Theorem 2] that $x \in \Psi_{\mathscr{I}_\omega}(E^c) \subset \Psi_{\mathscr{I}_\omega}(C^c)$. In particular, $\{x\} \cup C^c \in \mathscr{T}''_{\mathscr{I}_\omega}$ for every $x \in \mathbb{R}$. This implies that $Z^c \in \mathscr{T}''_{\mathscr{I}_\omega}$ for every set $Z \subset C$.

For the construction of the set P we use a nonempty perfect set $T \subset [\frac{1}{2}, 1]$ to be constructed later.

Put $t_n = e^n$ and define $P = \{0\} \cup \bigcup_{n\in\mathbb{N}} t_n^{-1}T$. Evidently, P is perfect. Also, $\limsup_{n\to\infty} \chi_{t_n P^c \cap (-1,1)} \le \chi_{(-1,1)\setminus T}$, as $T \subset t_n P$ for all $n \in \mathbb{N}$. So, $0 \notin \Psi_{\mathscr{I}_\omega}(P^c, \{t_n\})$; i.e., $\{0\} \cup P^c \notin \mathscr{T}''_{\mathscr{I}_\omega}$.

To prove that $\{0\} \cup P^c \in \mathscr{T}'_{\mathscr{I}_\omega}$ let us notice that it suffices to have the following condition:

$$k\, t_n^{-1}T \cap l\, t_m^{-1}T = \varnothing \quad \text{for all } k, l, m, n \in \mathbb{N},\ n \neq m. \tag{5}$$

This implies that $\{0\} \cup P^c \in \mathscr{T}'_{\mathscr{I}_\omega}$, because for every $x \in (-1, 1)$, $x \neq 0$, there is at most one number n such that $x \in \bigcup_{k \in \mathbb{N}} k\, t_n^{-1} T$. So, x belongs to at most finite number of sets $k\, P = \{0\} \cup \bigcup_{n \in \mathbb{N}} k\, t_n^{-1} T$, as for every $n \in \mathbb{N}$ there are at most finitely many k for which $x \in k\, t_n^{-1} T$.

To finish the proof we must find a perfect set $T \subset [\frac{1}{2}, 1]$ satisfying (5). We construct T as an intersection $T = \bigcap_{i \in \mathbb{N}} T_i$, where the sets T_i are defined by induction on i. Put $T_0 = [\frac{1}{2}, 1]$. Every set $T_i \subset T_{i-1}$, $i > 0$, is formed by 2^i pairwise disjoint closed intervals of equal length such that the set A_i of their end points contains only rational numbers. We construct the T_i from T_{i-1} by removing from every component interval $[a, b]$ of T_{i-1} an open interval (c, d), $a < c < d < b$, with the same center in such a way that

$$k\, t_n^{-1} T_i \cap l\, t_m T_i^{-1} = \varnothing \quad \text{for all } k, l, m, n < i, n \neq m.$$

But, the condition above is equivalent to the condition

$$T_i \cap \frac{k}{l} e^n T_i = \varnothing \quad \text{for all } k, l, n < i, n > 0. \tag{6}$$

Let B_{i-1} be the union of the sets $(k/l)e^n A_{i-1}$ for $k, l, n < i$, $n > 0$. Then $B_{i-1} \cap \mathbb{Q} = \varnothing$, as $A_{i-1} \subset \mathbb{Q}$ and e^n is irrational for every $n \in \mathbb{N}$, $n > 0$. Hence, $\varepsilon = \operatorname{dist}(A_{i-1}, B_{i-1}) > 0$. Let $\delta > 0$ be a number such that $(k/l)e^n \delta < \varepsilon/2$ for all $k, l, n < i$, and let us choose an interval $(c, d) \subset [a, b]$ in such a way that $b - d = c - a < \delta$. It is easy to see that this choice guarantees satisfaction of condition (6). □

We finish this paper with the following problem.

Problem 4. Is $\mathscr{T}''_{\mathscr{I}_\omega} \not\subset \mathscr{L}$?

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Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506

E-mail address: KCies@WVNVMS.bitnet

Department of Mathematics, University of Louisville, Louisville, Kentucky 40292

E-mail address: LMLars01@ULKYVX.CT.LOUISVILLE.edu