A RESULT ON MULTIDIMENSIONAL QUANTIZATION

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Abstract. For any integer $N > 1$, a probability space, a Gaussian random vector $X$ defined on the space with a positive definite covariance matrix, and an $N$-level quantizer $Q$ are presented such that the random vector $Q(X)$ takes on each of the $N$ values in its range with equal probability and such that $X$ and $Q(X)$ are independent.

Introduction

Quantization, the process by which a set is mapped into a finite subset of a given cardinality, plays a pivotal role in virtually any application that requires analog to digital conversion; indeed, it is at the heart of much of modern digital technology. In such applications, a quantizer is often taken to be a function mapping $\mathbb{R}^k$ into a subset of $\mathbb{R}^k$ of cardinality $N$, where $k$ is a positive integer and $N$ is an integer greater than one (see, e.g., [1, 5, 3, 6, 2, 7, 9]). In this paper we present what might be a surprising consequence of such a general approach to quantization.

Development

For a topological space $T$, we will let $\mathcal{B}(T)$ denote the family of Borel subsets of $T$. For a set $S$, we will let $\mathcal{P}(S)$ denote the power set of $S$ and $I_S$ denote the indicator function of $S$. By a standard Gaussian measure we will mean a Gaussian measure whose first moment is zero and whose second moment is one. Let $k$ be a positive integer. For any measure $m$ on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ we will let $m_*$ denote the inner measure on $(\mathbb{R}^k, \mathcal{P}(\mathbb{R}^k))$ induced by $m$ and we will let $m^*$ denote the outer measure on $(\mathbb{R}^k, \mathcal{P}(\mathbb{R}^k))$ induced by $m$. Recall from [4, p. 61] that if $B \in \mathcal{B}(\mathbb{R}^k)$ and $A \in \mathcal{P}(\mathbb{R}^k)$, then $m_*(B \cap A) + m^*(B \setminus A) = m(B)$. We will let $\lambda$ denote Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\Lambda$ denote Lebesgue measure on $(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ for integers $k > 1$, where $k$ will be determined from the context. Recall that for a measure space...
(\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k), m)$, a subset $S$ of $\mathbb{R}^k$ is said to be a saturated non-$m$-measurable set if $m_*(S) = m_*(S^c) = 0$. Finally, a $k$-dimensional quantizer of a random variable $X$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is any function $Q: \mathbb{R}^k \to F$ such that $F$ is a finite subset of $\mathbb{R}^k$, such that $Q(x) = x$ for all $x$ in $F$ (i.e., such that $Q$ restricted to $F$ is the identity map on $F$) and such that $Q(X)$ is itself a random variable defined on $(\Omega, \mathcal{F}, P)$. If $F$ is a finite subset of $\mathbb{R}^k$ with cardinality $N$ then a quantizer $Q: \mathbb{R}^k \to F$ of a random variable $X$ is said to be an $N$-level quantizer.

The following lemma is proved in [8, pp. 381–382].

**Lemma 1.** For any positive integer $M$ there exist $M$ disjoint subsets $Z_1, Z_2, \ldots, Z_M$ of the real line such that $Z_1, Z_2, \ldots, Z_M$ and $Z = Z_1 \cup \cdots \cup Z_M$ are saturated non-$\lambda$-measurable sets.

The next result is an immediate consequence of Lemma 1.

**Corollary 1.** For any integer $N > 1$ there exist $N$ subsets $T_1, T_2, \ldots, T_N$ of the real line that partition the real line and are such that for each positive integer $j \leq N$, $T_j$ is a saturated non-$\lambda$-measurable set.

For our purposes the following corollary will prove useful.

**Corollary 2.** For any positive integer $k$ and any integer $N > 1$, there exist $N$ subsets $S_1, S_2, \ldots, S_N$ of $\mathbb{R}^k$ that partition $\mathbb{R}^k$ and are such that, for each positive integer $j \leq N$, $S_j$ is a saturated non-$\lambda$-measurable set.

**Proof.** For $k = 1$, the result follows from Corollary 1. Assume $k > 1$. Let $T_1, \ldots, T_N$ be a partition of the real line as given by Corollary 1. For positive integers $j \leq N$, let $S_j = T_j \times \mathbb{R} \times \cdots \times \mathbb{R} \subset \mathbb{R}^k$. Fix a positive integer $j \leq N$ and assume that there exists an $\mathcal{F}_\sigma$ subset $B$ of $\mathbb{R}^k$ such that $B \subset S_j$ and $\Lambda(B) > 0$. Define a subset $\hat{B}$ of $\mathbb{R}$ as follows:

$$\hat{B} = \{b_1 \in \mathbb{R} : (b_1, b_2, \ldots, b_k) \in B \text{ for some } (b_2, \ldots, b_k) \in \mathbb{R}^{k-1}\}.$$ 

Note that $\hat{B} \in \mathcal{B}(\mathbb{R})$. Further, notice that $\lambda(\hat{B}) > 0$ since $B \subset \hat{B} \times \mathbb{R} \times \cdots \times \mathbb{R} \subset \mathbb{R}^k$ and $\Lambda(B) > 0$. But, $\lambda(\hat{B}) = 0$ since $\hat{B} \subset T_j$ and $\lambda_*(T_j) = 0$. This contradiction implies that $\Lambda(B) = 0$ and hence that $\Lambda_*(S_j) = 0$. It follows similarly that $\Lambda_*(S_j^c) = 0$ also. Q.E.D.

**Lemma 2.** For a positive integer $k$ and an integer $N > 1$, let $S_1, S_2, \ldots, S_N$ comprise a partition of $\mathbb{R}^k$ such that for each positive integer $j \leq N$, $S_j$ is a saturated non-$\lambda$-measurable set. The set

$$\mathcal{I} = \{(S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N) : A_i \in \mathcal{B}(\mathbb{R}^k) \text{ for } 1 \leq i \leq N\}$$

is a $\sigma$-algebra on $\mathbb{R}^k$.

**Proof.** Choosing $A_1 = \cdots = A_N = \emptyset$ implies that $\emptyset \in \mathcal{I}$. Let $A$ be an element of $\mathcal{I}$. Then $A = (S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N)$ for some choice of the $A_i$'s from $\mathcal{B}(\mathbb{R}^k)$. Further, $A^c = (S_1 \cap A_1)^c \cap \cdots \cap (S_N \cap A_N)^c$. Since

$$S_i^c = \bigcup_{j=1}^{N} S_j \quad (i \neq j)$$
it follows that

\[ A^c = \bigcap_{i=1}^{N} \bigcup_{j=1, j \neq i}^{N} S_j \cup A_i^c. \]

Hence \( A^c \) is a finite union of sets, each of which is of one of the following three forms:

(i) \( S_{n_1} \cap \cdots \cap S_{n_k} \cap B \) where \( 1 \leq n_1 < \cdots < n_k \leq N, \ k > 1, \) and \( B \in \mathcal{B}(\mathbb{R}^k); \)

(ii) \( S_j \cap B \) for \( 1 \leq j \leq N \) and \( B \in \mathcal{B}(\mathbb{R}^k); \)

(iii) \( B \in \mathcal{B}(\mathbb{R}^k). \)

Every set of the form given by (i) is empty since the \( S_i \)'s are disjoint. Further, any set \( B \in \mathcal{B}(\mathbb{R}^k) \) may be expressed as \( B = (S_1 \cap B) \cup \cdots \cup (S_N \cap B). \) Hence, \( A^c \) is an element of \( \mathcal{F}. \)

Finally, if \( B_1, B_2, \ldots \) are in \( \mathcal{F}, \) then for some choice of the \( A_{i,j} \)'s from \( \mathcal{B}(\mathbb{R}^k), \)

\[ \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty \bigcap_{j=1}^N (S_j \cap A_{i,j}) = \bigcap_{j=1}^N \left( \bigcup_{i=1}^\infty A_{i,j} \right) \in \mathcal{F}. \quad \text{Q.E.D.} \]

Recall that two measures \( P_1 \) and \( P_2 \) on a given measurable space \( (\Omega, \mathcal{F}) \) are said to be equivalent if \( \{ A \in \mathcal{F} : P_1(A) = 0 \} = \{ A \in \mathcal{F} : P_2(A) = 0 \}. \) Notice that for sets \( S_1, S_2, \ldots, S_N \) as above, it follows that, for any positive integer \( i \leq N \) and any \( \mathcal{B}(\mathbb{R}^k) \)-measurable set \( H, \ P_*(S_i \cap H) = 0, \ P_*(S_i^c \cap H) = 0, \ P^*(S_i \cap H) = P(H), \) and \( P^*(S_i^c \cap H) = P(H) \) for any probability measure \( P \) on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \) that is equivalent to Lebesgue measure on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)). \) The following lemma will be used in the proof of a subsequent theorem.

**Lemma 3.** For a positive integer \( k \) and an integer \( N > 1, \) let \( S_1, S_2, \ldots, S_N \) comprise a partition of \( \mathbb{R}^k \) such that for each positive integer \( j \leq N, \) \( S_j \) is a saturated non-\( \Lambda \)-measurable set. Let \( P \) be a probability measure on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \) that is equivalent to Lebesgue measure on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)). \) Let \( A_1, A_2, \ldots, A_N \) and \( B_1, \ldots, B_N \) be sets from \( \mathcal{B}(\mathbb{R}^k) \) such that

\[ (S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N) = (S_1 \cap B_1) \cup \cdots \cup (S_N \cap B_N). \]

Then \( P(A_i \triangle B_i) = 0 \) for any positive integer \( i \leq N \) where for any two subsets \( A \) and \( B \) of \( \mathbb{R}^k, \ A \triangle B \) denotes the symmetric difference of \( A \) and \( B. \)

**Proof.** Fix a positive integer \( i \leq N. \) By assumption,

\[ (S_1 \cap A_1) \cup \cdots \cup (S_N \cap A_N) = (S_1 \cap B_1) \cup \cdots \cup (S_N \cap B_N). \]

Intersecting each side with \( S_i \) implies that \( (S_i \cap A_1) \cup \cdots \cup (S_i \cap A_N) = (S_i \cap B_1) \cup \cdots \cup (S_i \cap B_N), \) which implies that \( (S_i \cap A_1) \cap (S_i \cap B_1)^c = (S_i \cap A_1) \cap (S_i \cap B_i)^c = (S_i \cap A_1) \cup (S_i \cap B_i)^c = (S_i \cap A_i \cap B_i)^c = 0 \) and, similarly, that \( (S_i \cap B_1) \cup (S_i \cap A_i)^c = 0. \) Thus, we see that \( (S_i \cap A_1) \cup (S_i \cap B_1) = S_i \cap (A_i \triangle B_i) = 0. \) Since \( (A_i \triangle B_i) \in \mathcal{B}(\mathbb{R}^k), \) it follows that \( P(A_i \triangle B_i) = P^*(S_i \cap (A_i \triangle B_i)) = P^*(0) = 0. \) \( \text{Q.E.D.} \)

The following theorem provides a probability space upon which the principal result of this paper will be based.

**Theorem 1.** For a positive integer \( k \) and an integer \( N > 1, \) let \( S_1, S_2, \ldots, S_N \) comprise a partition of \( \mathbb{R}^k \) such that for each positive integer \( j \leq N, \) \( S_j \) is a saturated non-\( \Lambda \)-measurable set. Let \( P \) be a probability measure on \( (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k)) \)
that is equivalent to Lebesgue measure on \((\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))\). There exists a probability space \((\mathbb{R}^k, \mathcal{G}, \mu)\) such that \(\mathcal{G}\) includes \(\mathcal{B}(\mathbb{R}^k)\), such that \(\mathcal{G}\) contains \(S_1, \ldots, S_N\), such that the measure \(\mu\) agrees with \(P\) on \(\mathcal{B}(\mathbb{R}^k)\), and such that \(\mathcal{B}(\mathbb{R}^k)\) is independent of \(\sigma(S_1, \ldots, S_N)\).

**Proof.** Let \(\mathcal{G}\) be the \(\sigma\)-algebra \(\mathcal{F}\) provided by Lemma 2. Recall that \(\mathcal{G}\) contains all sets of the form \((S_1 \cap A_1) \cup \cdots \cup (S_n \cap A_n)\) where \(A_i \in \mathcal{B}(\mathbb{R}^k)\) for each positive integer \(i \leq N\). If \(A \in \mathcal{B}(\mathbb{R}^k)\) then choosing \(A_1 = \cdots = A_N = A\) implies that \(A \in \mathcal{G}\). Similarly, for any positive integer \(i \leq N\), setting \(A_i = \mathbb{R}^k\) and all other \(A_j\)'s equal to the empty set implies that \(S_i \in \mathcal{G}\). Define a measure \(\mu\) on the measurable space \((\mathbb{R}^k, \mathcal{G})\) via

\[
\mu((S_1 \cap A_1) \cup \cdots \cup (S_n \cap A_n)) = \frac{1}{N} (P(A_1) + \cdots + P(A_N))
\]

for \((S_1 \cap A_1) \cup \cdots \cup (S_n \cap A_n) \in \mathcal{G}\). That \(\mu\) is well defined follows from Lemma 3 and that \(\mu\) is in fact a probability measure that agrees with \(P\) on \(\mathcal{B}(\mathbb{R}^k)\) is then straightforward. Further notice that \(\mu(S_i) = 1/N\) for each positive integer \(i \leq N\) and that, for any set \(B \in \mathcal{B}(\mathbb{R}^k)\) and any positive integer \(i \leq N\), \(\mu(S_i \cap B) = \frac{1}{N} P(B) = \mu(S_i) \mu(B)\). Thus \(S_i\) is independent of \(\mathcal{B}(\mathbb{R}^k)\) for each positive integer \(i \leq N\). Finally, notice that \(\mathcal{B}(\mathbb{R}^k)\) is in fact independent of \(\sigma(S_1, \ldots, S_N)\) since \(\{\emptyset, S_1, \ldots, S_N\}\) is a \(\pi\)-system. Q.E.D.

We are now in a position to state and prove the principal result of this paper.

**Theorem 2.** Let \(k\) be a positive integer and let \(N\) be an integer greater than one. There exists a probability space \((\Omega, \mathcal{F}, \nu)\), a Gaussian random vector \(X\) defined on \((\Omega, \mathcal{F}, \nu)\) taking values in \(\mathbb{R}^k\) with a positive definite covariance matrix, and an \(N\)-level \(k\)-dimensional quantizer \(Q: \mathbb{R}^k \rightarrow F\) such that \(\nu(Q(X) = x) = 1/N\) for each \(x\) in \(F\) and such that \(X\) and \(Q(X)\) are independent.

**Proof.** Let \(S_1, \ldots, S_N\) be sets as provided by Corollary 2. For these \(N\) subsets of \(\mathbb{R}^k\), let \((\Omega, \mathcal{F}, \nu)\) be a probability space as provided by Theorem 1 where \(P\) is chosen to be the product measure induced by placing standard Gaussian measure on each factor of \((\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))\). For each positive integer \(i \leq N\), let \(\alpha_i\) be an element from \(S_i\). Let \(F\) denote the set \(\{\alpha_1, \ldots, \alpha_N\}\). Define an \(N\)-level \(k\)-dimensional quantizer \(Q: \mathbb{R}^k \rightarrow F\) via \(Q(x) = \sum_{i=1}^{N} \alpha_i I_{S_i}(x)\). Further, notice that the random vector \(X(\omega) = \omega; \omega \in \Omega\), is a zero mean Gaussian random vector defined on \((\Omega, \mathcal{F}, \nu)\) whose covariance matrix is the \(k \times k\) identity matrix. Also, notice that for \(1 \leq i \leq N\), \(\nu(Q(X(\omega)) = \alpha_i) = \nu(\omega \in S_i) = 1/N\). Finally, notice that \(X\) and \(Q(X)\) are independent via Theorem 1. Q.E.D.

**REFERENCES**


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