A NOTE ON THE JACOBSON RADICAL

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Abstract. In this paper we solve a problem of Divinsky to represent the Jacobson radical as a lower radical class.

The purpose of this paper is to represent the Jacobson radical as a lower radical class.

For each class of rings $M$, there exists a radical class $U(M)$, called the upper radical class determined by $M$, such that $U(M)$ is the largest radical class with every ring in $M$ $U(M)$-semisimple [2, 5, 8]. For each class of rings $L$, there exists a radical class $Y(L)$, called the lower radical class determined by $L$, such that $Y(L)$ is the smallest radical class containing $L$ [1, 3, 4, 6–8]. It is clear that every radical class is the lower radical class determined by itself; the Jacobson radical can be characterized as the upper radical of the class of all primitive rings [1, 7, 8]. Divinsky raised the problem of identifying the Jacobson radical as the lower radical of some reasonable class of rings [1, 8]. We have the following solution.

Theorem 1. Let $J$ be the Jacobson radical class. For each ring $A$, let $J(A)$ be the Jacobson radical of $A$ and $J_2 = \{(J(A))^2 : A$ is a ring$\}$. Let $N$ be the class of all nilpotent rings. Then the Jacobson radical class is the lower radical class determined by $N$ and $J_2$.

Proof. It is clear that $Y(J_2 \cup N) \subseteq J$. Let $A \in J$, $I \not\subseteq A$ and $A/I$ be any nonzero quotient ring of $A$. If $I \supseteq (J(A))^2$, then $A \supseteq (J(A))^2$ and $A/I = (A/(J(A))^2)/(I/(J(A))^2)$. We have $(A/(J(A))^2)^2 = (J(A)/(J(A))^2)^2 = 0$. Hence $A/(J(A))^2 \in N$ and $0 \neq A/I \in N$.

If $I \not\supseteq (J(A))^2$ then $A/I \supseteq I + (J(A))^2/I \cong (J(A))^2/I \cap (J(A))^2 \neq 0$.

$$(J(A))^2/I \cap (J(A))^2 = (J(A)/I \cap (J(A))^2)^2$$
$$= [J(A/I \cap (J(A))^2)]^2 \in J_2.$$ Hence we have that $A \in Y(J_2 \cup N)$.

Let $S$ be any radical class and $S_n = \{(S(A))^n : A$ is a ring$\}$. By the same proof, we have the following more general result.
Theorem 2. If $S$ is a radical class containing all nilpotent rings $N$, then $S$ is the lower radical class determined by $N$ and $S_n$ for any positive integer $n$.

We wish to give an example to show that $N \not\subseteq J_2$.

Example. Let $A$ be the ring generated by $\{a\}$ with $2a = a^3 = 0$. Then $A$ is a nilpotent ring. If $A \in J_2$ then there is a ring $B \in J$ with $A = B^2$ and $B$ is nilpotent. Suppose that there exists $b$ in $B$ such that the ideal $C$ generated by $\{A, b\}$ satisfies $C^2 = A$. Then $b^2 \neq a$; otherwise, $b^3 = ab \notin A$. Also, $ba \neq a$, $ba \notin a^2 + a$, $ab \neq a$, $ba \notin a^2 + a$, $ba \neq a^2 + a$, $a^2b \neq a$, and $a^2b \neq a^2 + a$. For if $a = ba$ then $a = b^na = 0$. The other inequalities can be verified in a similar way. Hence $b^2 = a^2 + a$. Now $ba \neq 0$; otherwise, $0 = b^2a = a^3 + a^2 = a^2$. Hence $ba = a^2$ and $b^2 = a^2 + a = ba + a$, $a = b(b - a)$, and $a^2 = b(b - a)a = b(ba - a^2) = 0$. Hence $C^2 \not\subseteq A$ and $C^2 = \{0, a^2\}$.

Hence there are $b$ and $c$ in $B$ such that the ideal $D$ generated by $\{A, b, c\}$ satisfies $D^2 = A$, while the square of the ideal generated by $A$ and $b$ or $c$ is $\{0, a^2\}$. If $bc = a$ then $ca = a^2$; otherwise, $ca = 0$ and $0 = bca = a^2$. However, $a^2 = bca = ba^2 = b^2a^2 = b^na^2 = 0$ and hence $bc = a + a^2$. Then $ca = a^2$; otherwise, $ca = 0$, $a^2 = (a + a^2)a = bca = 0$. However, $a^2 = (a + a^2)a = bca = ba^2 = b^na^2 = 0$; which is a contradiction. Similarly $cb \neq a$ and $cb \neq a^2 + a$. Hence $D^2 \not\subseteq A$ and $B^2 \not\subseteq A$ and $A$ is not in $J_2$. Therefore $N \not\subseteq J_2$ and $J_2 \not\subseteq J$.

This example also shows that $N \not\subseteq S_n$ and $S_n \neq S$ for any radical class $S$ containing all nilpotent rings and $n \geq 2$. Because of the lack of a mechanism, it is tedious to verify a ring not in $J_2$. It would be interesting to find more examples of rings in $J \setminus J_2$ and to establish general properties.

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References

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