AN INVARIANT OF LINKS IN A HANDLEBODY ASSOCIATED WITH THE SPIN j REPRESENTATION OF $U_q(\mathfrak{sl}(2, \mathbb{C}))$

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ABSTRACT. We construct an invariant of framed links in a handlebody by means of the spin j representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$. We can see this invariant is an extension of the Jones polynomial and Kauffman's Dubrovnik polynomial. Moreover, we can obtain a linear representation of the generalized braid group associated with the Lie algebras of types B and C by applying the operators used in constructing an invariant to tangles in a solid torus.

INTRODUCTION

In [12] Turaev constructed an invariant of oriented links embedded in a solid torus by means of the vector representation of $U_q(\mathfrak{sl}(m, \mathbb{C}))$. We extend his method to links in a handlebody and construct an invariant of framed links in a handlebody by means of the spin j representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$.

Let D be the generic diagram obtained by projecting a framed link L in a handlebody $\Sigma_n \times [0, 1]$ of genus n onto Σ_n, where Σ_n is an n-holed disk. By a height function on D, we associate to each noncritical level a tensor product of some V_j's, where V_j is the representation space of the spin j representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$. Then, to each strip including a critical point, we define a linear operator derived from the tensor representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$. Furthermore, we introduce another operator that includes the parameters $\{t_i^{(l)}\}_{i,l}$ depending on the embedding of a link into a handlebody. Let $P_j(L)$ denote the image of $1 \in \mathbb{C}$ by the composition of these operators. We shall show that $P_j(L)$ is an ambient isotopy invariant of a framed link (cf. [5] about this notation) L embedded in a handlebody and that its value is in

$$\Lambda_n := \mathbb{Z}\,[q^{\pm 1},\, \{(t_j^{(l)})^{\pm 1},\, (t_{j-1}^{(l)})^{\pm 1},\, \ldots,\, (t_{-j}^{(l)})^{\pm 1}\}_{l=1,2,\ldots,n}]/\Im_n$$

where $\Im_n$ is the ideal generated by $\{t_{-i}^{(l)}\,(t_i^{(l)})^{-1} - 1,\ t_{i_1}^{(l)}\,t_{i_2}^{(l)} - t_{i_1+\gamma}^{(l)}\,t_{i_2-\gamma}^{(l)}\}_{1 \le l \le n}$ and n is the genus of the handlebody. Moreover, $P_j(L)$ has the following properties:

(1) $P_j(\bigcirc) = [2j + 1]$;

(2) $P_j(\text{Ȣ}) = (-1)^{2j} q^{2j(j+1)} P_j(\cup)$,
    $P_j(\text{ɤ}) = (-1)^{2j} q^{-2j(j+1)} P_j(\cup)$.

Ignoring the parameters $\{t_i^{(l)}\}_{i,l}$ related to the embedding of a link into a handlebody, by the definition of $P_j(L)$, we have invariants of links in S^3. Especially, in the case of $j = \frac{1}{2}$, we shall observe that our invariant is a certain extension of the Jones polynomial $V_L(s) \in \mathbb{Z}[s^{\pm 1/2}]$. Namely, put

$$\widetilde{P}(L; q) := \frac{(-q^{3/2})^{-w(D)}}{[2]} P_{1/2}(L);$$

then we have

$$\widetilde{P}(L; s^{1/2}) = V_L(s) \quad \text{for an oriented link } L \text{ in } S^3.$$

Similarly, in the case of $j = 1$, put

$$\widehat{P}(\overline{L}; q) := \frac{1}{[3]} P_1(\overline{L}),$$

and let $\mathscr{D}_{\overline{L}}(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ be Kauffman's Dobrouvnik polynomial, then we have

$$\widehat{P}(\overline{L}; q) = \mathscr{D}_{\overline{L}}(q^4, q^2 - q^{-2}) \quad \text{for an unoriented link } \overline{L} \text{ in } S^3.$$

Moreover, as an application, we can obtain a linear representation of the generalized braid group associated with the Lie algebras of types B and C by applying the operators used in constructing an invariant to tangles in a solid torus.

1. $U_q(\mathfrak{sl}(2, \mathbb{C}))$ AND ITS SPIN j REPRESENTATION

In this section, we summarize the basic properties derived from $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and its spin j representation. Let us recall that $U_q(\mathfrak{sl}(2, \mathbb{C}))$, the quantized universal enveloping algebra of Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in the sense of Drinfel'd [2] and Jimbo [3], is the associative $\mathbb{C}$-algebra generated by $\{1, k^{\pm 1}, X^+, X^-\}$ with relations

$$k X^{\pm} k^{-1} = q^{\pm 1} X^{\pm}, \qquad X^+ X^- - X^- X^+ = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

where $q \in \mathbb{C} \backslash \{0, \pm 1\}$.

The finite-dimensional irreducible representations of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ are parametrized by half-integers $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ [8, 10]. This representation (ϱ_j, V_j) is called the spin j representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$, where $\dim_{\mathbb{C}} V_j = 2j + 1$. The actions of the generators of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ with respect to the basis $\{e_m^j\}_{-j \le m \le j}$ of V_j is given as

$$\varrho_j (X^{\pm}) e_m^j = ([j \mp m] [j \pm m + 1])^{1/2} e_{m \pm 1}^j,$$
$$\varrho_j (k) e_m^j = q^m e_m^j$$

where $[n] := (q^n - q^{-n})/(q - q^{-1})$.

It is well known that there exist an invertible element R in the completed tensor product $U_q(\mathfrak{sl}(2, \mathbb{C})) \,\hat{\otimes}\, U_q(\mathfrak{sl}(2, \mathbb{C}))$, which is called the universal R-matrix, such that $\check{R}^{j_1 j_2} := T \circ ((\varrho_{j_1} \otimes \varrho_{j_2})\, R)$ satisfies the Yang-Baxter equation

$$
(1.1) \quad
\begin{aligned}
(\check{R}^{j_1 j_2} \otimes \mathrm{Id}_{V_{j_3}}) \circ (\mathrm{Id}_{V_{j_2}} \otimes \check{R}^{j_1 j_3}) \circ (\check{R}^{j_2 j_3} \otimes \mathrm{Id}_{V_{j_1}}) \\
= (\mathrm{Id}_{V_{j_1}} \otimes \check{R}^{j_2 j_3}) \circ (\check{R}^{j_1 j_3} \otimes \mathrm{Id}_{V_{j_2}}) \circ (\mathrm{Id}_{V_{j_3}} \otimes \check{R}^{j_1 j_2})
\end{aligned}
$$

where T is defined by

$$
T\left(e^{j_1}_{m_1} \otimes e^{j_2}_{m2}\right) = e^{j_2}_{m_2} \otimes e^{j_1}_{m_1}
$$

for the basis of $V_{j_1} \otimes V_{j_2}$ [2, 3]. We can express $\check{R}^{j_1 j_2}$ explicitly with respect to the basis of $V_{j_1} \otimes V_{j_2}$ as [7, 11]

$$
(1.2) \quad \check{R}^{j_1 j_2}\left(e^{j_1}_{m_1} \otimes e^{j_2}_{m_2}\right) = \sum_{n \geq 0} (R^{j_1 j_2})^{m_1+n, m_2-n}_{m_1, m_2} \left(e^{j_2}_{m_2-n} \otimes e^{j_1}_{m_1+n}\right);
$$

here the nonzero element $(R^{j_1 j_2})^{m_1+n, m_2-n}_{m_1, m_2}$ is given by

$$
(R^{j_1 j_2})^{m_1+n, m_2-n}_{m_1, m_2}
$$

$$
\begin{aligned}
:= &\frac{(1 - q^{-2})^n}{[n]!} \left(\frac{[j_1 - m_1]!\,[j_1 + m_1 + n]!\,[j_2 + m_2]!\,[j_2 - m_2 + n]!}{[j_1 - m_1 - n]!\,[j_1 + m_1]!\,[j_2 + m_2 - n]!\,[j_2 - m_2]!}\right)^{1/2} \\
&\times q^{n\,(m_1 - m_2 + 2n) + 2\,(m_1 + n)\,(m_2 - n)}
\end{aligned}
$$

where $[n]! := [n]\,[n-1]\cdots[1]$, $[0]! := 1$. Moreover

$$
(1.3) \quad (\check{R}^{j_1 j_2^{-1}})^{m_2+n, m_1-n}_{m_1, m_2} = (-1)^n q^{-(2m_2+n)\,(2m_1-n)+n} (R^{j_1 j_2})^{m_2+n, m_1-n}_{m_2, m_1}
$$

where $(\check{R}^{(j_1 j_2^{-1})})^{m_2+n, m_1-n}_{m_1, m_2}$ is a nonzero matrix element for $(\check{R}^{j_1 j_2})^{-1}$.

If q is not a root of unity then the tensor representation $V_{j_1} \otimes V_{j_2}$ is decomposable into the direct sum of irreducible representations [3], that is,

$$
V_{j_1} \otimes V_{j_2} = \bigoplus_{\substack{|j_1-j_2| \leq i \leq j_1+j_2 \\ j_1+j_2+i \in \mathbb{Z}}} V_i.
$$

The projection map $Y^{j_1 j_2}_i : V_{j_1} \otimes V_{j_2} \to V_i$ is a $U_q(\mathfrak{sl}(2, \mathbb{C}))$-module homomorphism and is uniquely determined up to scalar multiple. Moreover, put

$$
Y^i_{j_1 j_2} := {}^t Y^{j_1 j_2}_i : V_i \to V_{j_1} \otimes V_{j_2}.
$$

Then we have the following orthogonality relation between $Y^{j_1 j_2}_i$ and $Y^i_{j_1 j_2}$ with an appropriate normalization [7, 9]

$$
(1.4) \qquad (1) \qquad \sum_{\substack{|j_1-j_2| \leq i \leq j_1+j_2 \\ j_1+j_2+i \in \mathbb{Z}}} Y^i_{j_1 j_2} \circ Y^{j_1 j_2}_i = \mathrm{Id}_{V_{j_1} \otimes V_{j_2}},
$$

$$
(1.5) \qquad (2) \qquad Y^{j_1 j_2}_k \circ Y^l_{j_1 j_2} = \delta_{k, l}\, \mathrm{Id}_{V_k}.
$$

Let $X^{j_1 j_2}_i := Y^i_{j_1 j_2} \circ Y^{j_1 j_2}_i : V_{j_1} \otimes V_{j_2} \to V_{j_2} \otimes V_{j_1}$. Then we have a spectral decomposition of $\check{R}^{j_1 j_2}$ as [4]

$$
(1.6) \qquad (\check{R}^{j_1 j_2})^{\pm 1} = \sum_{\substack{|j_1-j_2| \leq i \leq j_1+j_2 \\ j_1+j_2+i \in \mathbb{Z}}} (-1)^{j_1+j_2+i}\, q^{\pm\{\xi(i)-\xi(j_1)-\xi(j_2)\}/2} X^{j_1 j_2}_i
$$

where $\xi(i) = 2i(i+1)$.

2. Construction of Invariants

In this section, we construct invariants of links in a way analogous to that of Turaev [12].

Let Σ_n be a disk with n holes and $I = [0, 1]$. We regard a handlebody of genus n as $\Sigma_n \times I$. Suppose Σ_n is embedded in $\mathbb{R}^2$ and the coordinate of the center of Σ_n coincides $(0, 0)$. For convenience, we assume the holes $H_1, H_2, \ldots, H_n$ of Σ_n are arranged as shown in Figure 2.1 and the coordinates of their centers are $(h_1, 0), (h_2, 0), \ldots, (h_n, 0)$, respectively. Namely,

$$H_i = \{(x, y) \mid (x - h_i)^2 + y^2 \leq \varepsilon^2\} \quad \text{for } 1 \leq i \leq n,$$

$$\Sigma_n = \{(x, y) \mid x^2 + y^2 \leq r^2\} - \bigcup_{i=1}^{n} \{\operatorname{Int} H_i\}$$

where ε and r are the radii of H_i and Σ_n, respectively.

Let D be a link diagram in $\operatorname{Int} \Sigma_n$ that represents a framed link L in $\Sigma_n \times I$. Here we assume that the diagram D is taken such that the linking number of D and its push-off coincides with the given framing on L. Such a diagram is called a *good diagram* in terms of Kirby-Melvin [7]. Furthermore, we assume that the diagram D is taken to be generic with respect to the height function $x|_D$ (or to be in general position in terms of [12]).

A point of D is called critical if it is the point in which the tangent line is parallel to the y-axis or the self-crossing point of D (Figure 2.2).
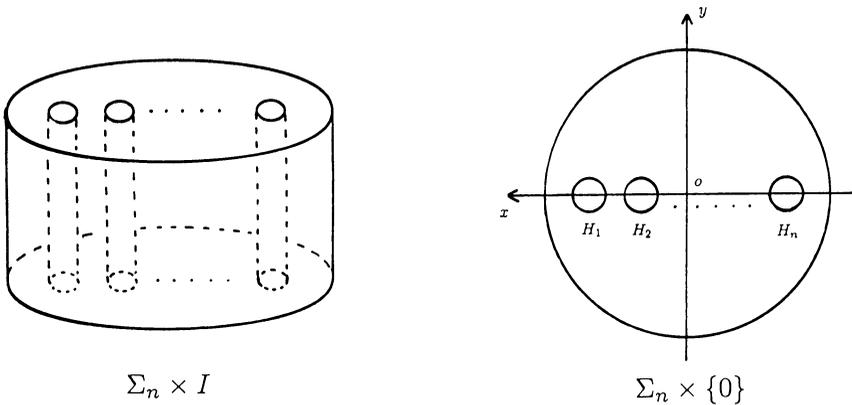

$$\Sigma_n \times I \qquad\qquad \Sigma_n \times \{0\}$$

FIGURE 2.1

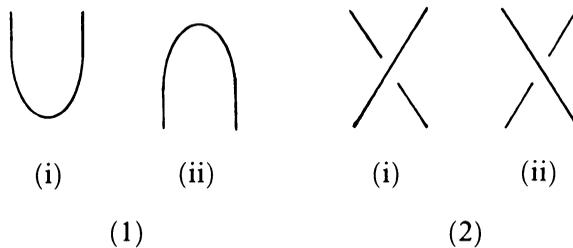(i)   (ii)   (i)   (ii)

(1)   (2)

FIGURE 2.2

Let d be the number of critical points of D. We draw $d + n + 1$ parallel lines $\{x = \alpha_i\}_{1 \le i \le d+n+1}$ satisfying the following conditions: (1) $-r = \alpha_1 < \alpha_2 < \cdots < \alpha_{d+n+1} = r$, (2) each open strip $\alpha_i < x < \alpha_{i+1}$ ($i = 1, 2, \ldots, d+n$) either contains exactly one critical point of D or does not contain critical points of D, (3) there exists i such that $\alpha_i < h_l < \alpha_{i+1}$ for all $l = 1, 2, \ldots, n$, (4) none of these lines $\{x = \alpha_i\}$ contains a critical point of D.

With each line $x = \alpha_i$, $i = 1, 2, \ldots, d+n$, we associate a complex vector space $W^i := V^{\otimes k}$ where V is the representation space of the spin j representation of $U_q(\mathfrak{sl}(2, \mathbb{C}))$ and k is the number of points in $\{x | x = \alpha_i\} \cap D$. In particular, if $k = 0$ then we define $W^i := \mathbb{C}$.

With each strip $\alpha_i < x < \alpha_{i+1}$ we associate a linear operator $f_i : W^i \to W^{i+1}$ in the following way.

Let $\{e_j, e_{j-1}, \ldots, e_{-j}\}$ be the basis of V. Assume $q \in \mathbb{C}$ is not a root of unity. With the critical points depicted in Figure 2.2 we associate the following operators respectively:

(1) (i) $c := \sqrt{[2j+1]}\, Y^0_{jj} : \mathbb{C} \to V \otimes V$,
$$c(1) = \sum_{i=-j}^{j} (-1)^{j-i} q^i e_i \otimes e_{-i};$$
(ii) $a := \sqrt{[2j+1]}\, Y^{jj}_0 : V \otimes V \to \mathbb{C}$,
$$a(e_k \otimes e_l) = (-1)^{j-k} q^k \delta_{k,-l}.$$
(2) (i) $\check{R} := \check{R}^{jj} : V \otimes V \to V \otimes V$;
(ii) $\check{R}^{-1} := (\check{R}^{jj})^{-1} : V \otimes V \to V \otimes V$

where Y^0_{jj}, Y^{jj}_0, and $\check{R}^{jj}$ are defined as in §1 with $j_1 = j_2 = j$. With the strand parallel to the x-axis we associate the identity operator Id_V. Then, for the strip containing a critical point we define a linear operator $f_i : W^i \to W^{i+1}$ by

$$f_i := \mathrm{Id}_V \otimes \cdots \otimes \mathrm{Id}_V \otimes f \otimes \mathrm{Id}_V \otimes \cdots \otimes \mathrm{Id}_V$$

where $f \in \{a, c, \check{R}, \check{R}^{-1}\}$.

Furthermore, for the strip $\alpha_i < h_l < \alpha_{i+1}$, with the strand across the segment $\{(h_l, y) | y \le 0\} \cap \Sigma_n$ we associate Id_V and with the strand across the segment $\{(h_l, y) | y \ge 0\} \cap \Sigma_n$ we associate the operator

$$\theta_l : V \to V, \qquad \theta_l(e_i) = q^{x^{(l)}_i} t^{(l)}_i e_i \quad \text{for } i = j, j-1, \ldots, -j$$

where we assume that $x^{(l)}_i \in \mathbb{Z}$ and $t^{(l)}_i \in \mathbb{C}^\times$ satisfy

$$x^{(l)}_{-i} = -x^{(l)}_i, \qquad x^{(l)}_{i_1} + x^{(l)}_{i_2} = x^{(l)}_{i_1+\gamma} + x^{(l)}_{i_2-\gamma},$$
$$t^{(l)}_{-i} = (t^{(l)}_i)^{-1}, \qquad t^{(l)}_{i_1} t^{(l)}_{i_2} = t^{(l)}_{i_1+\gamma} t^{(l)}_{i_2-\gamma}$$

for $0 \le \gamma \le \min(j - i_1, j + i_2)$, respectively. Then, for the strip $\alpha_i < h_l < \alpha_{i+1}$ (see Figure 2.3 on the next page) we define a linear operator $f_i : W^i \to W^{i+1}$ by

$$f_i := \underbrace{\mathrm{Id}_V \otimes \cdots \otimes \mathrm{Id}_V}_{u \text{ times}} \otimes \underbrace{\theta_l \otimes \cdots \otimes \theta_l}_{v \text{ times}}.$$

Since $W^i = W^{d+n+1} = \mathbb{C}$, we have a linear operator

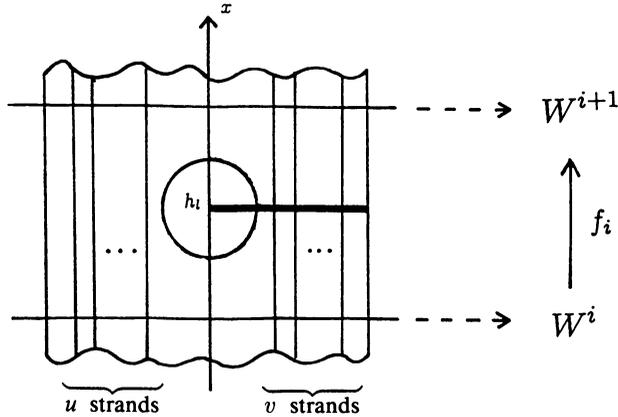$$P^{\chi_n}_D := f_{d+n} \circ f_{d+n-1} \circ \cdots \circ f_1 : \mathbb{C} \to \mathbb{C}$$

FIGURE 2.3

where $\chi_n := \{x_j^{(1)}, \ldots, x_{-j}^{(1)}, x_j^{(2)}, \ldots, x_{-j}^{(n)}\}$. Suppose q and $t_i^{(l)}$ ($-j \leq i \leq j$, $1 \leq l \leq n$) are indeterminates of $\mathbb{C}$, then, by definition, $P_D^{\chi_n}(1)$ has its value in

$$\Lambda_n := \mathbb{Z}\,[q^{\pm 1}, \{(t_j^{(l)})^{\pm 1}, (t_{j-1}^{(l)})^{\pm 1}, \ldots, (t_{-j}^{(l)})^{\pm 1}\}_{l=1,2,\ldots,n}]/\Im_n$$

where $\Im_n$ is the ideal generated by $\{t_{-i}^{(l)}\,(t_i^{(l)})^{-1} - 1,\ t_{i_1}^{(l)}\,t_{i_2}^{(l)} - t_{i_1+\gamma}^{(l)}\,t_{i_2-\gamma}^{(l)}\}_{1 \leq l \leq n}$.

We assert that $P_D^{\chi_n}(1)$ is an invariant of links in a handlebody as follows.

Theorem 2.1. *Let* $\chi_n = \{x_j^{(1)}, \ldots, x_{-j}^{(1)}, x_j^{(2)}, \ldots, x_{-j}^{(n)}\} \subset \mathbb{Z}$ *be fixed. Then* $P_j(L) := P_D^{\chi_n}(1) \in \Lambda_n$ *is an ambient isotopy invariant of a framed link* L *embedded in a handlebody of genus* n *and has the following properties*:

    (1) $P_j(\bigcirc) = [2j + 1]$;

    (2) $P_j(\gamma) = (-1)^{2j} q^{2j(j+1)} P_j(\cup)$,

          $P_j(\zeta) = (-1)^{2j} q^{-2j(j+1)} P_j(\cup)$.

Before proving Theorem 2.1, we note the relationships among the operators. They will give the proof of the first half of Theorem 2.1.

Lemma 2.1. (1) $(\mathrm{Id}_V \otimes a) \circ (c \otimes \mathrm{Id}_V) = \mathrm{Id}_V = (a \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes c)$.

    (2) $(\mathrm{Id}_V \otimes a) \circ (\check{R} \otimes \mathrm{Id}_V) = (a \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \check{R}^{-1})$,

        $(\mathrm{Id}_V \otimes a) \circ (\check{R}^{-1} \otimes \mathrm{Id}_V) = (a \otimes \mathrm{Id}_V) \circ (\mathrm{Id}_V \otimes \check{R})$.

Lemma 2.2. (1) $(\theta_l \otimes \theta_l) \circ c = c$.

    (2) $a \circ (\theta_l \otimes \theta_l) = a$.

    (3) $\check{R} \circ (\theta_l \otimes \theta_l) = (\theta_l \otimes \theta_l) \circ \check{R}$,

        $\check{R}^{-1} \circ (\theta_l \otimes \theta_l) = (\theta_l \otimes \theta_l) \circ \check{R}^{-1}$.

The proofs of Lemmas 2.1 and 2.2 are given by the direct calculations from the definitions of the operators.

Proof of Theorem 2.1. As is shown in [9, 12, 13], to prove the first half of the theorem, it is sufficient to verify that each operator corresponding to the diagrams in Figures 2.4 and 2.5 is invariant under the associated modification illustrated in the figures. In fact, they are assured by Lemmas 2.1 and 2.2 and the properties of the R-matrix.
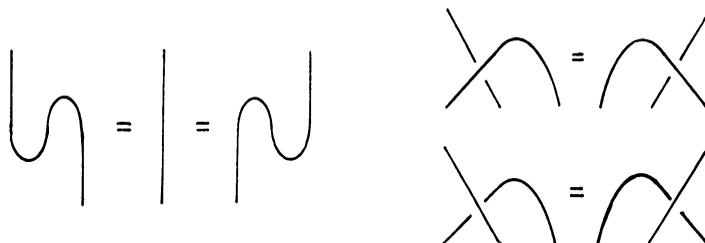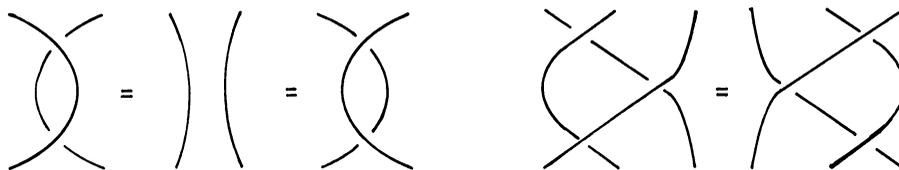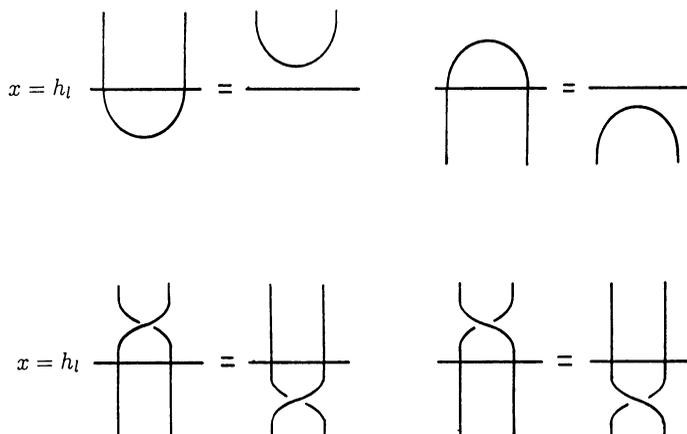
FIGURE 2.4

FIGURE 2.5

The properties of $P_j(L)$ can be easily checked as follows:

(1) By definition,

$$P_j(\bigcirc) = a \circ c(1) = \sum_{i=-j}^{j} q^{2i} = [2j+1].$$

(2) By (1.4), (1.5), (1.6), and the definition of c,

$$\check{R}^{-1} \circ c = \sqrt{[2j+1]}\check{R}^{-1} \circ Y_{jj}^0$$
$$= \sqrt{[2j+1]}(-1)^{2j}q^{2j(j+1)}Y_{jj}^0 = (-1)^{2j}q^{2j(j+1)}c.$$

This implies the first formula. The second formula is also obtained in a similar way. □

Remark. It is clear that if $L = L_1 \cup L_2$ is a split link in a solid torus then $P_j(L) = P_j(L_1) \cdot P_j(L_2)$.

3. EXAMPLES

3.1. The case of $j = \frac{1}{2}$.

Corollary 3.1. *For an oriented link L in a handlebody of genus n, put*

$$\widetilde{P}(L) := \frac{(-q^{3/2})^{-w(D)}}{[2]} P_{1/2}(L)$$

where D is a diagram of L and $w(D)$ is the writhe of D. Then $\widetilde{P}(L)$ is an ambient isotopy invariant of L such that

(1) $\widetilde{P}(\bigcirc) = 1$;
(2) $q^2 \widetilde{P}(\overset{\curvearrowright}{\times}) - q^{-2} \widetilde{P}(\overset{\curvearrowleft}{\times}) = (q^{-1} - q)\widetilde{P}(\smallsmile\hspace{-4pt}\frown)$.

Proof. (2) is obtained by (1.4), (1.5), and (1.6). The other assertions are clear. □

Thus, by the definition of the invariant, we have

Corollary 3.2. *Let $\chi_n = \{0, \dots, 0\}$ and $\widetilde{P}(L; q) := \widetilde{P}(L)$. If $t_i^{(k)} = 1$ for $i = -\frac{1}{2}, \frac{1}{2}$ and $k = 1, 2, \dots, n$, then $\widetilde{P}(L; s^{1/2}) = V_L(s)$ where $V_L(s) \in \mathbb{Z}[s^{\pm 1/2}]$ is the Jones polynomial of an oriented link L in S^3.*

3.2. The case of $j = 1$. Similarly, we obtain the following corollaries.

Corollary 3.3. *For an unoriented link $\overline{L}$ in a handlebody of genus n, put*

$$\widehat{P}(\overline{L}) := \frac{1}{[3]} P_1(\overline{L}).$$

Then $\widehat{P}(\overline{L})$ is a regular isotopy invariant of $\overline{L}$ such that

(1) $\widehat{P}(\bigcirc) = 1$;
(2) $\widehat{P}(\overset{\diagup}{\diagdown}) - \widehat{P}(\overset{\diagdown}{\diagup}) = (q^2 - q^{-2})[\widehat{P}(\smile\hspace{-3pt}\frown) - \widehat{P}(\asymp)]$;
(3) $\widehat{P}(\overset{\curvearrowright}{\gamma}) = q^4 \widehat{P}(\cup)$, $\widehat{P}(\overset{\curvearrowleft}{\gamma}) = q^{-4} \widehat{P}(\cup)$.

Remark. Hence $(q^4)^{-w(D)}\widehat{P}(\overline{L})$ is an ambient isotopy invariant for an oriented link L in a handlebody when $\widehat{P}(\overline{L})$ is defined on oriented link diagrams by forgetting the orientation.

Corollary 3.4. *Let $\chi_n = \{0, \dots, 0\}$ and $\widehat{P}(\overline{L}; q) := \widehat{P}(\overline{L})$. If $t_i^{(k)} = 1$ for $i = -1, 0, 1$ and $k = 1, 2, \dots, n$, then*

$$\widehat{P}(\overline{L}; q) = \mathscr{D}_{\overline{L}}(q^4, q^2 - q^{-2})$$

where $\mathscr{D}_{\overline{L}}(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is Kauffman's Dubrovnik polynomial for an unoriented link $\overline{L}$ in S^3.

4. A REPRESENTATION OF THE GENERALIZED BRAID GROUP ASSOCIATED WITH THE LIE ALGEBRAS OF TYPES B AND C

Let us recall the definition of the generalized braid group due to Brieskorn [1]. Let $\mathscr{H}_s$ be a hyperplane in $\mathbb{R}^n$. Let W be a linear transformation group of
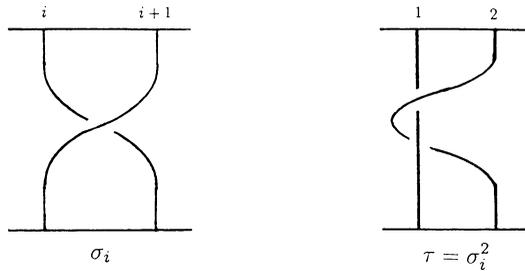
FIGURE 4.1

$\mathbb{R}^n$ generated by reflections with respect to $\mathcal{H}_s$'s. We denote by s the reflection with respect to $\mathcal{H}_s$. For a connected component of $\mathbb{R}^n - \bigcup_s \mathcal{H}_s$, there exists a set $\{\mathcal{H}_s\}_{s \in S}$ such that (W, S) becomes a Coxeter system; namely, W has a presentation with a generating set S and a relation $(ss')^{m(s, s')} = 1$ where $m(s, s')$ is the order of ss' and $m(s, s) = 1$. Let $\mathcal{H}_s^{\mathbb{C}}$ be the complexification of $\mathcal{H}_s$ in $\mathbb{C}^n$. The action of W is naturally extended to $\mathbb{C}^n - \bigcup_s \mathcal{H}_s^{\mathbb{C}}$. In [1] Brieskorn defined the generalized braid group as the fundamental group $\pi_1((\mathbb{C}^n - \bigcup_s \mathcal{H}_s^{\mathbb{C}})/W, *)$ and he proved that the generalized braid group associated with W has a presentation with generators $\{g_s\}_{s \in S}$ and a relation

$$(g_s g_t)^{m(s, t)} = (g_t g_s)^{m(s, t)}.$$

By investigating the root systems of the Lie algebras of types B and C, we can show that the generalized braid group G obtained from the Coxeter group associated with these Lie algebras has a presentation with the generators $\{h, g_2, g_3, \ldots, g_n\}$ and the relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for} \quad i = 2, 3, \ldots, n,$$
$$g_i g_j = g_j g_i \quad \quad \text{for} \quad |i - j| \geq 2,$$
$$h g_i = g_i h \quad \quad \text{for} \quad i = 3, 4, \ldots, n,$$
$$h g_2 h g_2 = g_2 h g_2 h.$$

Let B_{n+1} be the $(n+1)$-string braid group with the generators $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, n,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \quad \text{for} \quad |i - j| \geq 2.$$

By replacing the generator σ_1 with $\tau := \sigma_1^2$, we can define the subgroup $B_{n+1}^1 \subset B_{n+1}$. Namely, any element of B_{n+1}^1 has the fixed first string (Figure 4.1). We can verify that there is an isomorphism $B_{n+1}^1 \to G$ that transforms τ to h and σ_i to g_i for $i = 2, 3, \ldots, n$.

Now we embed the elements of B_{n+1}^1 in a solid torus $\Sigma_1 \times I$ by regarding the fixed first string as $H_1 \times I$ as in Figure 4.2. We consider the diagrams of these embedded elements of B_{n+1}^1. The diagrams corresponding to the generators of B_{n+1}^1 are illustrated in Figure 4.3, where $\times$ indicates the fixed first string.
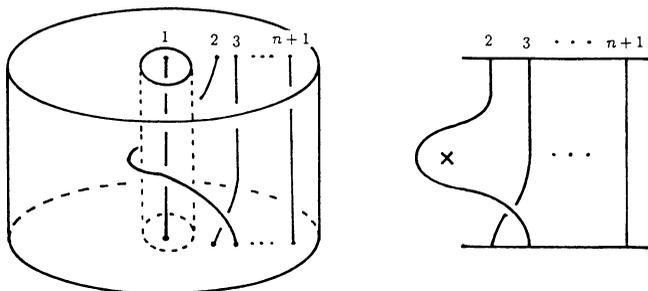
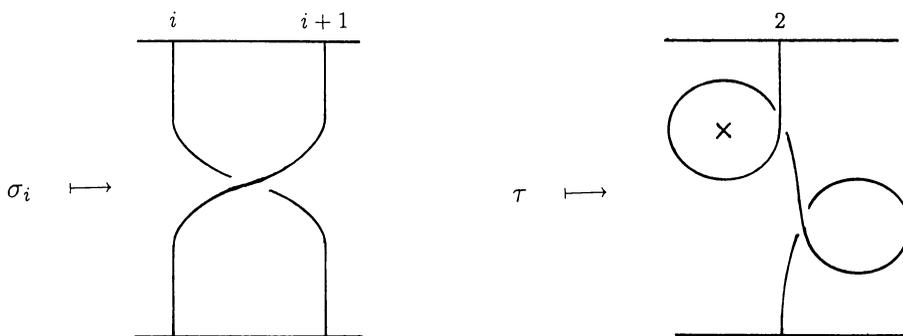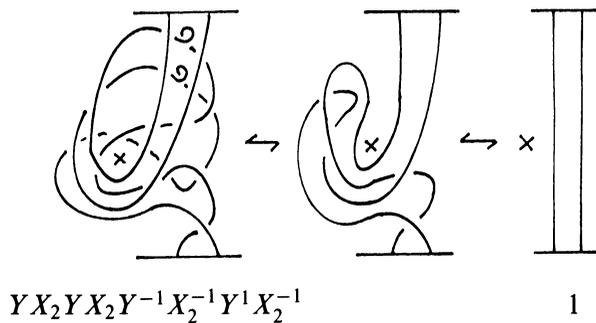$$YX_2YX_2Y^{-1}X_2^{-1}Y^1X_2^{-1} \qquad\qquad 1$$

FIGURE 4.4

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