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# LINKS AND NONSHELLABLE CELL PARTITIONINGS OF $S^3$

STEVE ARMENTROUT

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ABSTRACT. A cell partitioning of a closed 3-manifold $M^3$ is a finite covering of $M^3$ by 3-cells that fit together in a bricklike pattern. A cell partitioning $H$ of $M^3$ is shellable if $H$ has a counting $\langle h_1, h_2, \ldots, h_n\rangle$ such that if $1 \le i < n$, $h_1 \cup h_2 \cup \cdots \cup h_i$ is a 3-cell. The main result of this paper is a relationship between nonshellability of a cell partitioning $H$ of $S^3$ and the existence of links in $S^3$ specially related to $H$. This result is used to construct a nonshellable cell partitioning of $S^3$.

A *cell partitioning* of a closed 3-manifold $M^3$ is a finite covering $H$ of $M^3$ by 3-cells such that if $n$ is a positive integer greater than one and $n$ distinct 3-cells of $H$ have a point in common, then their common part is a cell of dimension $4-n$, where sets of negative dimension are empty. Thus the 3-cells of $H$ fit together in a bricklike pattern.

Cell partitionings of 3-manifolds provide an alternative to triangulations among combinatorial structures on the manifold. Cell partitionings may be thought of as dual to triangulations. In some way, cell partitionings seem simpler than triangulations, since their 2-skeleta have an essentially canonical structure.

Bing proved in [2, Theorem 2] that each closed 3-manifold has a cell partitioning. (Our use of the term "partitioning" differs from that of [2].) The main result of [2, Theorem 1] is a characterization of $S^3$ in terms of partitionings (in the sense of [2]) and a notion Bing later called sequential connectivity [3]. This latter idea is related to the notion of shellability.

A cell partitioning $H$ of a closed 3-manifold is *shellable* if and only if $H$ has a counting $\langle h_1, h_2, \ldots, h_n\rangle$ such that if $1 \le i < n$ then $h_1 \cup h_2 \cup \cdots \cup h_i$ is a 3-cell. A counting of $H$ with the indicated property is a *shelling* of $H$. We can characterize $S^3$ among closed 3-manifolds by using this idea: a closed 3-manifold is a 3-sphere if and only if it has a shellable cell partitioning.

We may define cell partitionings of compact 3-manifolds with boundary, and then a version of shellability can be used to characterize 3-cells among such 3-manifolds.

We may adapt the definitions above to define *disc partitionings* of closed 2-manifolds and compact 2-manifolds with boundary. There are definitions of shellability appropriate to such contexts. Then by using theorems of plane

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topology, we may prove that every disc partitioning of either $S^2$ or a disc is shellable [2, Theorem 10].

In contrast, $S^3$ has nonshellable cell partitionings. In this paper, we shall construct such partitionings using links in $S^3$. For a construction using knots in $S^3$, see [1]. For an example of a nonshellable cell partitioning of a 3-cell, see [3].

One may define shellability in terms of triangulations [3]. See [4] for an example of a nonshellable rectilinear triangulation of a rectilinear 3-simplex.

By a straightforward coning argument, we may prove that for each cell partitioning $H$ of a closed 3-manifold $M^3$, there is a triangulation of $M^3$ relative to which $H$ is polyhedral.

The main results of this paper deal with some connections between nonshellability of cell partitionings of closed 3-manifolds and the existence of certain links in the 3-manifold. Since we shall study linking, we shall work generally in homology 3-spheres.

## 1. The link property

In this section, we shall develop a criterion for nonshellability of cell partitionings of a closed 3-manifold based on linking pairs of disjoint simple closed curves in the manifold. This can be formulated in homology 3-spheres.

Suppose $M^3$ is a triangulated homology 3-sphere. Then a *2-component link* in $M^3$ in the union of two mutually disjoint polygonal simple closed curves in $M^3$. Two such curves $k$ and $l$ are *homologically linked* if and only if for each orientable polyhedral surface $S$ in $M^3$ bounded by $k$ and in general position relative to $l$, the intersection number (using integral coefficients) is nonzero. This notion is well defined and is independent of both the surface $S$ and the order in which the curves are named. The statement that $(k, l)$ *is a 2-component link* means that the link is $k \cup l$ and that $k$ and $l$ are the two simple closed curves forming the link. A 2-component link $(k, l)$ is *homologically nontrivial* if and only if $k$ and $l$ are homologically linked.

Suppose $M^3$ is a triangulated homology 3-sphere. A polyhedral cell partitioning $H$ of $M^3$ has the *link property* if and only if for each connected ordering $\langle h_1, h_2, \dots, h_n\rangle$ of $H$, there exist an integer $j$ with $1 < j < n$ and a homologically nontrivial 2-component link $(k, l)$ in $M^3$ such that $k \subset h_1 \cup h_2 \cup \cdots \cup h_j$ and $l \subset h_{j+1} \cup \cdots \cup h_n$. An ordering $\langle h_1, h_2, \dots, h_n\rangle$ of $H$ is *connected* if and only if for each $i$, $h_1 \cup h_2 \cup \cdots \cup h_i$ is connected.

**Proposition 1.** *If a polyhedral cell partitioning $H$ of a triangulated 3-sphere has the link property then $H$ is not shellable.*

*Proof.* Suppose $H$ is a polyhedral cell partitioning of a triangulated homology 3-sphere. Suppose $H$ has the link property and is shellable. Let $\langle h_1, h_2, \dots, h_n\rangle$ be a shelling of $H$. This ordering is connected. Then since $H$ has the link property, there exist an integer $j$ with $1 < j < n$ and a 2-component link $(k, l)$ in $M^3$ such that $k \subset h_1 \cup h_2 \cup \cdots \cup h_j$ and $l \subset h_{j+1} \cup \cdots \cup h_n$. But since $\langle h_1, h_2, \dots, h_n\rangle$ is a shelling of $H$, $h_1 \cup h_2 \cup \cdots \cup h_j$ is a 3-cell. It follows that $k$ bounds an orientable polyhedral surface $S$ in $h_1 \cup h_2 \cup \cdots \cup h_j$ and disjoint from $l$. Thus $k$ and $l$ are not homologically linked in $M^3$, and this is a contradiction. Hence $H$ is not shellable. □

## 2. Homology 3-spheres distinct from $S^3$

**Theorem 2.** *A homology 3-sphere $M^3$ is topologically distinct from $S^3$ if and only if every cell partitioning of $M^3$ has the link property.*

*Proof.* Suppose $M^3$ is a homology 3-sphere topologically distinct from $S^3$ and that $H$ is a cell partitioning of $M^3$. We may assume that $M^3$ is triangulated and $H$ is polyhedral.

By the results of [2], $H$ is not sequentially connected. Let $\langle h_1, h_2, \dots, h_n\rangle$ be any connected counting of $H$. Then there exists some integer $j$ such that $1 < j < n$ and $(h_1 \cup h_2 \cup \cdots \cup h_{j-1}) \cap h_j$ is not connected. Let $\Delta = (h_1 \cup h_2 \cup \cdots \cup h_{j-1}) \cap h_j$. Each component of $\Delta$ is a punctured disc.

Let $\omega$ be a point of $\operatorname{Bd} h_j$ not in $\Delta$. There is a component $D$ of $\Delta$ such that if $\gamma$ is the outer (with respect to $\omega$) boundary curve of $\Delta$, then the component $U$ of $(\operatorname{Bd} h_j) - \gamma$ not containing $\omega$ intersects no component of $\Delta$ other than $D$. Let $\widehat{D} = \overline{U}$; $\widehat{D}$ is a disc. Then since $\Delta$ is not connected, there is a component $E$ of $\Delta$ distinct from $D$. Note that $\widehat{D}$ and $E$ are disjoint.

Let $p$ be a point of $\operatorname{Int} D$, and let $q$ be a point of $\operatorname{Int} E$. Let $\lambda$ be a polygonal simple closed curve that is the union of two arcs $\alpha$ and $\beta$, where (1) $\alpha$ is an arc spanning $h_j$ from $p$ to $q$ and (2) $\beta$ is an arc spanning $h_1 \cup h_2 \cup \cdots \cup h_{j-1}$ from $p$ to $q$.

Let $\mu = \gamma$. Then $\lambda \subset h_1 \cup h_2 \cup \cdots \cup h_j$ and $\mu \subset h_{j+1} \cup \cdots \cup h_n$ since $\mu \subset \operatorname{Bd}(h_1 \cup h_2 \cup \cdots \cup h_j)$. Further, $\lambda$ and $\mu$ are homologically linked in $M^3$ since $\widehat{D}$ is a disc in $M^3$ with $\mu$ as its boundary and that contains exactly one point, a piercing point, of $\lambda$.

Hence $H$ has the link property. □

## 3. A nonshellable cell partitioning of $S^3$

In this section, we shall use the ideas of this paper to construct a nonshellable cell partitioning of $S^3$. We shall do this by constructing a cell partitioning of $S^3$ with the link property.

Let $C_6$ be the complete graph on six vertices.

**Lemma 3.** *There is a piecewise linear embedding $g\colon C_6 \to S^3$ such that any two disjoint simple closed curves in $g(C_6)$ are homologically linked in $S^3$.*

*Proof.* Let $J_1, J_2, \dots, J_{20}$ be the distinct simple closed curves in $C_6$. If $1 \le i \le 20$, let $K_i$ denote the simple closed curve in $C_6$ disjoint from $J_i$.

Let $g_1\colon C_6 \to S^3$ be a linear embedding of $C_6$ in $S^3$. If $j = 1, 2, \dots,$ or 20, let $I_j$ be a straight interval joining a point of $g_1(J_j)$ to a point of $g_1(K_j)$, these points not being vertices of $g_1(C_6)$. We may make the construction of the $I$'s and adjust $g_1(C_6)$ slightly if necessary, so that the $I$'s are mutually disjoint and have only their end points on $g_1(C_6)$. For $j = 1, 2, \dots, 20$, thicken $I_j$ slightly to obtain a cylinder $I_j^*$.

For each simple closed curve $J$ in $C_6$, orient $g_1(J)$. There are exactly 10 (unordered) pairs of disjoint simple closed curves in $g_1(C_6)$, and let $n_1, n_2, \dots, n_{10}$ be the homological linking numbers of these pairs.

Modify $g_1(J_1)$ in $I_1^*$ as shown in Figure 1 so that the homological linking number of $g_1(K_1)$ and the modified $g_1(J_1)$ is an integer $m_1$ chosen so that no sum of any subset of $\{\pm n_1, \pm n_2, \dots, \pm n_{10}, \pm m_1\}$ is zero. This yields a

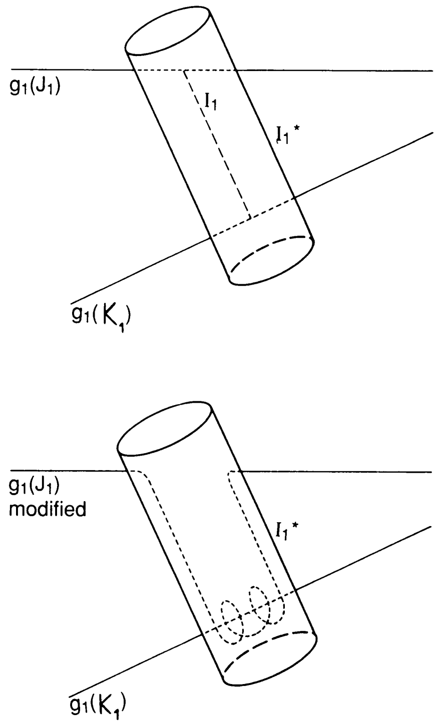


FIGURE 1

modification of $g_1: C_6 \to S^3$ in $I_1^*$ only, and let $g_2: C_6 \to S^3$ denote this modification. Orient the $g_2(J)$'s in an obvious manner.

Suppose $1 < k \le 20$ and we have modified the embeddings on $J_1, J_2, \ldots, J_{k-1}$ in a manner analogous to that described above, obtaining a piecewise linear embedding $g_k: C_6 \to S^3$ and integers $m_1, m_2, \ldots, m_{k-1}$ such that no sum of any subset of $\{\pm n_1, \pm n_2, \ldots, \pm n_{10}, \pm m_1, \ldots, \pm m_{k-1}\}$ is zero. We then modify: $C_6 \to S^3$ in $I_k^*$ only, as suggested by Figure 1 so that the homological linking number of $g_k(K_k)$ and the modified $g_k(J_k)$ is an integer $m_k$ chosen so that no sum of any subset of $\{\pm n_1, \pm n_2, \ldots, \pm n_{10}, \pm m_1, \ldots, \pm m_{k-1}, \pm m_k\}$ is zero. This yields a modification $g_{k+1}: C_6 \to S^3$.

Let $g: C_6 \to S^3$ denote the final such modified piecewise linear embedding $g_{21}: C_6 \to S^3$.

Now we shall show that every two disjoint simple closed curves in $g(C_6)$ are homologically linked. Suppose $J_u$ and $J_v$ are disjoint simple closed curves in $C_6$. It is clear that the homological linking number of $g(J_u)$ and $g(J_v)$ is the sum of some subsets of $\{\pm n_1, \pm n_2, \ldots, \pm n_{10}, \pm m_1, \ldots, \pm m_{20}\}$. Since, by construction, this sum is nonzero, $g(J_u)$ and $g(J_v)$ are homologically linked. □

We shall conclude this paper by using Lemma 3 to construct a cell partitioning $H$ of $S^3$ with the link property.

**Lemma 4.** *There is a cell partitioning $H$ of $S^3$ such that $H$ has the link property.*

*Proof.* By Lemma 3, there is a piecewise linear embedding $g: C_6 \to S^3$ of the complete graph $C_6$ on six vertices into $S^3$ such that any two disjoint simple closed curves in $g(C_6)$ are homologically linked. Let $v_1, v_2, \dots, v_6$ be the vertices of $C_6$. Let $N$ be a thin polyhedral tubular neighborhood of $g(C_6)$. Divide $N$ into six 3-cells by using polyhedral discs, one through a midpoint of each edge of $g(C_6)$ and transverse to that edge. If $i = 1, 2, \dots$ or 6, let $E_i$ denote the one of the resulting 3-cells containing $g(v_i)$.

It is easy to construct a cell partitioning $H$ of $S^3$ having $E_1, E_2, \dots, E_6$ among the 3-cells of $H$.

We shall now prove that $H$ has the link property. Suppose that $\langle h_1, h_2, \dots, h_n\rangle$ is a shelling of $H$.

Let $E_{i_1}$, $E_{i_2}$, and $E_{i_3}$ be the first three of $E_1, E_2, \dots, E_6$ in the shelling of $H$, and let $E_{i_4}$, $E_{i_5}$, and $E_{i_6}$ be the remaining three. Let $J$ be the simple closed curve in $C_6$ with vertices $v_{i_1}$, $v_{i_2}$, and $v_{i_3}$, and let $K$ be the simple closed curve in $C_6$ with vertices $v_{i_4}$, $v_{i_5}$, and $v_{i_6}$. Then $g(J) \subset E_{i_1} \cup E_{i_2} \cup E_{i_3}$ and $g(K) \subset E_{i_4} \cup E_{i_5} \cup E_{i_6}$.

Let $j$ be the integer such that $C_{i_3} = h_j$. Then $1 < j < n$, $C_{i_1} \cup C_{i_2} \cup C_{i_3} \subset h_1 \cup h_2 \cup \cdots \cup h_j$, and $C_{i_4} \cup C_{i_5} \cup C_{i_6} \subset h_{j+1} \cup \cdots \cup h_n$. It follows that $g(J) \subset \langle h_1, h_2, \dots, h_j\rangle$ and $g(K) \subset h_{j+1} \cup \cdots \cup h_n$.

Now $J$ and $K$ are disjoint, and by Lemma 12, $g(J)$ and $g(K)$ are homologically linked. It follows that $H$ has the link property. $\square$

**Corollary 5.** *$H$ is a nonshellable cell partitioning of $S^3$.*

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DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802-6401