INVERSIONS OF HERMITE SEMIGROUP

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Abstract. Let \( \{e^{-cH}|c \geq 0\} \) be the Hermite semigroup on the real line \( \mathbb{R} \). Then a representation is constructed for inversions of the semigroup, and it gives a representation of \( e^{-cH} \) for \( c < 0 \). Moreover, some characterizations of the domain in which, for \( c < 0 \), \( e^{-cH} \) is well defined are examined.

1. Introduction

We let \( \mu \) be the Gaussian measure \( \frac{1}{\sqrt{\pi}}e^{-x^2}dx \) on \( \mathbb{R} \). Then the family of the normalized Hermite polynomials
\[
h_n(x) = \frac{1}{\sqrt{2^n n!}} e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots,
\]
is a complete orthonormal system in \( L^2(\mu) \). For any \( f \in L^2(\mu) \), let \( f(x) = \sum_{n=0}^{\infty} a_n h_n(x) \). If \( c \) is nonnegative, then the series \( \sum_{n=0}^{\infty} a_n e^{-cn} h_n(x) \) converges in \( L^2(\mu) \). Hence we can define the linear operator \( e^{-cH} \) on \( L^2(\mu) \) by
\[
(e^{-cH}f)(x) = \sum_{n=0}^{\infty} a_n e^{-cn} h_n(x),
\]
and the operator norm of \( e^{-cH} \) is 1. The Hermite semigroup on \( \mathbb{R} \) means the family \( \{e^{-cH}|c \geq 0\} \). More generally, for every complex number \( c \) with \( \Re c > 0 \) or \( c = 0 \), we shall consider the operator \( e^{-cH} \), which is examined in several papers (for instance, see \([4, 10]\)). For \( \Re c > 0 \), the operator \( e^{-cH} \) is represented by
\[
(e^{-cH}f)(x) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} p_c(x-\xi) f(\xi) e^{-\xi^2} d\xi, \quad f \in L^2(\mu), \ x \in \mathbb{R},
\]
where the integral kernel
\[
p_c(x-\xi) = \sum_{n=0}^{\infty} e^{-cn} h_n(x) h_n(\xi) = (1 - e^{-2c})^{-1/2} \exp \left\{ \xi^2 - \frac{(\xi - e^{-c}x)^2}{1 - e^{-2c}} \right\}.
\]
In this paper, for any \( c \) with \( \Re c > 0 \) we shall give a representation for the inverse of the operator \( e^{-cH} \) in terms of integrals. Our argument for its representation is directly related to an extension of the operator \( e^{-cH} \) for \( \Re c < 0 \). When \( \Re c < 0 \), it is obvious that the domain in which the operator \( e^{-cH} \) is well defined is properly contained in \( L^2(\mu) \). We let \( \mathcal{D}(e^{-cH}) = \{ f \in L^2(\mu) \mid e^{-cH}f \in L^2(\mu) \} \). Then we give two characterizations of the members in \( \mathcal{D}(e^{-cH}) \) in terms of analytic extensibility of members in \( L^2(\mu) \) as entire functions.

We shall use the general method for integral transforms ([8] or [9, p. 82]) and some ideas for best approximation oriented in [1] on reproducing kernel Hilbert spaces.

2. INVERSE OF \( e^{-cH} \)

For any fixed complex number \( c \) with \( \Re c > 0 \), applying the dominated convergence theorem and Morera's theorem to the integral representation (1.2), we see that every member in the range of the operator \( e^{-cH} \) can be analytically extended to the complex plane \( \mathbb{C} \). Hence we shall consider the operator \( e^{-cH} \) as the linear operator of \( L^2(\mu) \) into an entire function space. Then, following the method of characterizing the ranges of integral transforms ([8] or [9, p. 82]), we obtain

**Theorem 1.1.** For any \( c \) with \( \Re c > 0 \), the range of \( L^2(\mu) \) under the operator \( e^{-cH} \) coincides with the Hilbert space consisting of entire functions with finite norms

\[
\| g \|_c^2 = \frac{2|w|^2}{\pi \sqrt{1-|w|^4}} \cdot \int_c |g(z)|^2 \exp \left\{ -2|w|^2 \left( \frac{x^2}{1+|w|^2} + \frac{y^2}{1-|w|^2} \right) \right\} \, dx \, dy
\]

where \( z = x + iy \) and \( w \) denotes \( e^{-c} \). Moreover, the isometrical identity

\[
\| e^{-cH}f \|_c^2 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} |f(x)|^2 e^{-x^4} \, dx, \quad f \in L^2(\mu),
\]

is valid.

**Proof.** For any pair \( (z, u) \in \mathbb{C} \times \mathbb{C} \), we calculate the complex kernel form

\[
K(z, \bar{u}; c) = \frac{1}{\pi} \int_{\mathbb{R}} p_c(z - \xi)p_c(u - \bar{\xi})e^{-c^2 \xi^2} \, d\xi
\]

\[
= \frac{1}{\sqrt{\pi(1-w^2)(1-\bar{w}^2)}} \exp \left\{ -\frac{w^2z^2}{1-w^2} \right\} \exp \left\{ \frac{-\bar{w}^2\bar{u}^2}{1-\bar{w}^2} \right\}
\]

\[
\cdot \int_{\mathbb{R}} \exp \left\{ -\frac{1-|w|^4}{(1-w^2)(1-\bar{w}^2)} \xi^2 \right\} \exp \left\{ \frac{2(1-\bar{w}^2)wz + 2(1-w^2)\bar{w}u}{(1-w^2)(1-\bar{w}^2)} \xi \right\} \, d\xi
\]

\[
= \frac{1}{\sqrt{1-|w|^4}} \exp \left\{ -\frac{|w|^4z^2}{1-|w|^4} \right\} \exp \left\{ -\frac{|w|^4\bar{u}^2}{1-|w|^4} \right\} \exp \left\{ \frac{2|w|^2z\bar{u}}{1-|w|^4} \right\}.
\]
Since $K(z, u; c)$ is a positive matrix on $\mathbb{C}$, it uniquely determines the reproducing kernel Hilbert space $H_c$ admitting the reproducing kernel $K(z, u; c)$. From [8] or [9, p. 82], the space $H_c$ is the range of $L^2(\mu)$ under the operator $e^{-cH}$ and we have the isometrical identity

$$\|e^{-cH}f\|_{H_c}^2 = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} |f(x)|^2 e^{-x^2} \, dx, \quad f \in L^2(\mu),$$

because the family $\{p_c(z-\xi)|z \in \mathbb{C}\}$ is complete in $L^2(\mu)$. Hence it is sufficient to prove that the members in $H_c$ are characterized as entire functions with finite norms (2.1). If $g \in H_c$, then $g$ is expressible in the form

$$(2.3) \quad g(z) = (1-|w|^4)^{-1/2} \exp \left\{ \frac{-|w|^4 z^2}{1-|w|^4} \right\} g_1(z),$$

where $z \in \mathbb{C}$ and $g_1$ is a member in the reproducing kernel Hilbert space $H_{\exp\{2|w|^2 z \bar{w}/(1-|w|^4)\}}$ admitting the reproducing kernel $\exp\{2|w|^2 z \bar{w}/(1-|w|^4)\}$. Moreover, the following isometrical identity holds:

$$(2.4) \quad \|g\|_{H_c}^2 = (1-|w|^4)^{-1/2} \|g_1\|_{H_{\exp\{2|w|^2 z \bar{w}/(1-|w|^4)\}}}^2.$$

For these contents, see [2, p. 358]. Meanwhile, referring to [3, p. 198], we have

$$\|g_1\|_{H_{\exp\{2|w|^2 z \bar{w}/(1-|w|^4)\}}}^2 = \frac{2|w|^2}{\pi(1-|w|^4)} \int_{\mathbb{R}} |g_1(z)|^2 \exp \left\{ -\frac{2|w|^2 |z|^2}{1-|w|^4} \right\} \, dx \, dy.$$

Therefore, from (2.3) and (2.4) our theorem is proved.

For two complex numbers $c_1$, $c_2$ with $\Re c_1 > 0$ and $\Re c_2 > 0$, we shall discuss a relation between $H_{c_1}$ and $H_{c_2}$. Put $g_1 = e^{-c_1H}f$ and $g_2 = e^{-c_2H}f$ for some $f \in L^2(\mu)$. If $\Re c_1 > \Re c_2$, then $g_1 = e^{-c_1H}f = e^{-(c_1-c_2)H} e^{-c_2H}f = e^{-(c_1-c_2)H} g_2$, and so we can directly obtain a representation of $g_1$ in terms of $g_2$ by using the integral representation (1.2). However, if $\Re c_1 \leq \Re c_2$, it is not obvious to represent $g_1$ in terms of $g_2$. Hence we are interested in this case. Furthermore, we shall establish the inverse formula of the integral transform (1.2).

**Theorem 2.2.** For $\Re c_1 > 0$, $\Re c_2 > 0$, and $f \in L^2(\mu)$, let $g_1 = e^{-c_1H}f$ and $g_2 = e^{-c_2H}f$. Then $g_1$ is expressible in the form

$$(2.5) \quad g_1(\xi) = \frac{2|w_2|^2}{\pi \sqrt{(1-|w_2|^4)(1-w_1^2 w_2^2)}} \exp \left\{ \frac{-w_1^2 w_2^2 \xi^2}{1 - w_1^2 w_2^2} \right\} \cdot \int_{\mathbb{C}} g_2(z) \exp \left\{ \frac{w_1 w_2 \bar{z}}{1 - w_1^2 w_2^2} (2\xi - w_1 \bar{w}_2 \bar{z}) - 2|w_2|^2 \left( \frac{x^2}{1+|w_2|^2} + \frac{y^2}{1-|w_2|^2} \right) \right\} \, dx \, dy,$$

where $w_1 = e^{-c_1}$, $w_2 = e^{-c_2}$, and $g_2(z)$ is the analytic extension of $g_2$ to $\mathbb{C}$. 
Moreover, $f$ is given by

$$f(\xi) = \text{l.i.m.} \frac{2|w_2|^2}{\pi \sqrt{(1 - |w_2|^4)(1 - r^2|w_2|^2)}} \exp \left\{ -\frac{r|w_2|^2\xi^2}{1 - r^2|w_2|^2} \right\} \cdot \iint g_2(z) \exp \left\{ \frac{r|w_2|^2}{1 - r^2|w_2|^2} (2\xi - r\overline{w_2} \overline{z}) - 2|w_2|^2 \left( \frac{x^2}{1 + |w_2|^2} + \frac{y^2}{1 - |w_2|^2} \right) \right\} \, dx \, dy. \tag{2.6}$$

The notation l.i.m. means the $L^2(\mu)$-convergence.

**Proof.** From the definition (1.1) of $e^{-ciH}$ and the identity (2.5), the expression (2.6) is obvious. We assume that $T_{c_1}$ is the inverse operator of $e^{-ciH}$ from $H_{c_1}$ to $L^2(\mu)$. In addition, let $S_{c_1, c_2}$ be the linear operator of $H_{c_1}$ into $H_{c_2}$ defined by

$$S_{c_1, c_2}g = e^{-ciH}T_{c_1}g, \quad g \in H_{c_1}.$$  

Then we have $S_{c_1, c_2}g_1 = g_2$. Since the operator $S_{c_1, c_2}$ is an isometry of $H_{c_1}$ onto $H_{c_2}$, the adjoint operator $S_{c_1, c_2}^*$ of $S_{c_1, c_2}$ is the inverse of it. Hence, for $\xi \in \mathbb{R}$ we get the representation

$$g_1(\xi) = [S_{c_1, c_2}, g_2](\xi) = (S_{c_1, c_2}^* g_2, K(\cdot \xi; c_1))_{H_{c_1}} = (g_2, S_{c_1, c_2}K(\cdot \xi; c_1))_{H_{c_2}}.$$

Meanwhile, for $z \in \mathbb{C}$ the following is valid:

$$[S_{c_1, c_2}K(\cdot \xi; c_1)](z) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} p_{c_1}(z - x)p_{c_1}(\overline{\xi} - \overline{x})e^{-x^2} \, dx$$

$$= \frac{1}{\sqrt{\pi(1 - |w_1|^2)(1 - |w_2|^2)}} \exp \left\{ -\frac{|w_1|^2\xi^2}{1 - |w_1|^2} \right\} \exp \left\{ -\frac{|w_2|^2z^2}{1 - |w_2|^2} \right\} \cdot \int_{\mathbb{R}} \exp \left\{ -\frac{1 - |w_1|^2w_2^2}{(1 - |w_1|^2)(1 - |w_2|^2)} x^2 \right. \right.$$ 

$$+ \left. 2(1 - |w_2|^2)w_1\xi + 2(1 - |w_1|^2)w_2z \right\} \frac{x}{(1 - |w_1|^2)(1 - |w_2|^2)} \, dx$$

$$= \frac{1}{\sqrt{1 - |w_1|^2w_2^2}} \exp \left\{ -\frac{|w_1|^2w_2^2\xi^2}{1 - |w_1|^2|w_2|^2} \right\} \cdot \exp \left\{ -\frac{w_1^2w_2^2z^2}{1 - |w_1|^2|w_2|^2} \right\} \exp \left\{ \frac{2w_1w_2z\xi}{1 - |w_1|^2|w_2|^2} \right\}.$$

Therefore, we obtain the identity (2.5).

In the expression (2.6), $r$ can be taken on a path in $\{0 < |z| < 1\}$ with the terminal point 1, in general.

**Remark.** For any $n$, $h_n(z)$ denotes the analytic extension of $h_n(x)$ to $\mathbb{C}$. Then we see that, for $\Re c > 0$, the family $\{e^{-cn}h_n(z)|n = 0, 1, 2, \ldots\}$ is
a complete orthonormal system in $H_c$. Hence, the expression $p_c(z - \xi) = \sum_{n=0}^\infty e^{-cn}h_n(z)h_n(\xi)$ and Theorem 1.1 suggest the representation of $T_c$ in the form

$$[T_c g](\xi) = (g, p_c(z - \xi))_{H_c}, \quad g \in H_c.$$  

For fixed $\xi$, however, the integral in the right-hand side of (2.7) need not converge. In this case, in [3, p. 202; 7; 9, p. 85] the method of taking some exhaustions of $C$ was considered in a natural way. According to such a general method, we can obtain another representation of (2.6):

$$f(\xi) = \frac{2|w_2|^2}{\pi \sqrt{1 - |w_2|^4}} \cdot \int_{|z| \leq \sigma} g_2(z) p_c(z - \xi) \exp \left\{ -2|w_2|^2 \left( \frac{x^2}{1 + |w_2|^2} + \frac{y^2}{1 - |w_2|^2} \right) \right\} \, dx \, dy$$

$$= \frac{2|w_2|^2}{\pi \sqrt{(1 - |w_2|^4)(1 - w_2^2)}} \cdot \int_{|z| \leq \sigma} g_2(z) \exp \left\{ \frac{\xi^2 - (\xi - w_2)^2}{1 - w_2^2} - 2|w_2|^2 \left( \frac{x^2}{1 + |w_2|^2} + \frac{y^2}{1 - |w_2|^2} \right) \right\} \, dx \, dy.$$

3. Extension of $e^{-cH}$ for $\Re c < 0$

The inverse transform (2.6) means an extension of $e^{-cH}$ for $\Re c < 0$ because $f(x)$ has the representation $\sum_{n=0}^\infty a_n e^{-(-c)n}h_n(x)$ in $L^2(\mu)$ when $g_2(x) = \sum_{n=0}^\infty a_n h_n(x)$. Hence, for any $c$ with $\Re c < 0$, we define the linear operator $e^{-cH}$ in the form (1.1). Then, in the expression (2.6) replacing $w_2$ by $e^c$, we obtain the representation of $e^{-cH}$. However, since the expression (2.6) requires the analytic extension form of a member in $L^2(\mu)$, we shall give its representation in terms of real variable.

For any fixed $c$ with $\Re c < 0$, we first establish two characterizations of the members in $\mathcal{D}(e^{-cH})$. Since the family $\{e^{cn}h_n(z)\mid n = 0, 1, 2, \ldots\}$ is a complete orthonormal system in $H_{-c}$, for a given $f \in L^2(\mu)$, $f \in \mathcal{D}(e^{-cH})$ if and only if $f$ is almost everywhere equal to the restriction of a member in $H_{-c}$ to $\mathbb{R}$ with respect to $\mu$. In this connection, we recall the interesting result in [5, p. 166]:

For any $\alpha > 0$, let $h$ be an entire function satisfying $\int\int_C |h(z)|^2 e^{-y^2/\alpha} \, dx \, dy < \infty$. Then

$$\frac{1}{\sqrt{\pi \alpha}} \int_C |h(z)|^2 e^{-y^2/\alpha} \, dx \, dy = \sum_{n=0}^\infty \frac{\alpha^n}{n!} \int_R \left| \frac{d^n}{dx^n} h(x) \right|^2 \, dx.$$  

Conversely, if $h(x)$ is a $C^\infty$-function on $\mathbb{R}$ with a convergent sum in the right-hand side of (3.1), then $h$ can be analytically extended to $\mathbb{C}$ and satisfies the identity (3.1).

This result will contribute to our result in the following way.
Theorem 3.1. For any \( c \) with \( \Re c < 0 \), let \( w = e^c \). If \( f \) is a \( \mathcal{C}^\infty \)-function in \( L^2(\mu) \), then the following are equivalent:

1. \( f \) is a member in \( \mathcal{D}(e^{-cH}) \).
2. The series

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{1 - |w|^4}{4|w|^2} \right)^n \int_{\mathbb{R}} \left| \frac{d^n}{dx^n} \left[ f(x) \exp \left\{ -\frac{|w|^2 x^2}{1 + |w|^2} \right\} \right] \right|^2 dx
\]

converges.

3. The integral

\[
\int_{\mathbb{C}} \left| \frac{f(\xi)}{1 - |w|^4} \right|^2 d\xi \cdot \exp \left[ \frac{-2|w|^4}{(1 - |w|^4)(|w|^8 + |w|^4 + 1)} \cdot \{(|w|^4 + |w|^2 + 1)x^2 - (|w|^4 - |w|^2 + 1)y^2\} \right] dx dy
\]

is finite.

Proof. If \( f \) is the restriction of a member \( \hat{f} \) in \( H_{-c} \) to \( \mathbb{R} \), then we have

\[
\|\hat{f}\|_{-c}^2 = \frac{2|w|^2}{\pi \sqrt{1 - |w|^4}} \int_{\mathbb{C}} \left| \frac{f(z)}{1 - |w|^4} \right|^2 \exp \left\{ -\frac{|w|^2 z^2}{1 + |w|^2} \right\} \exp \left\{ -4|w|^2 y^2 \right\} dx dy.
\]

Therefore, from the identity (3.1) the two statements (1) and (2) are equivalent.

Next, we prove that \( f \) is the restriction of a member in \( H_{-c} \) to \( \mathbb{R} \) if and only if \( f \) satisfies the condition (3). For simplicity, let \( T \) be the restriction operator to \( \mathbb{R} \). Then \( T \) is a bounded operator of \( H_{-c} \) into \( L^2(\mu) \) (in fact, the operator norm \( \|T\| \) is 1), and the adjoint operator \( T^* \) of \( T \) has the following representation:

\[
[T^*h](z) = (h, TK(\cdot, \bar{z}; -c))_{L^2(\mu)}, \quad z \in \mathbb{C},
\]

for all \( h \in L^2(\mu) \). Since the range \( T(H_{-c}) \) is dense in \( L^2(\mu) \), the operator \( T^* \) is one-to-one. Hence, \( f \) can be extended as the member in \( H_{-c} \) if and only if \( T^*f \) is contained in the range \( T^*T(H_{-c}) \). If \( g \) is a member in \( T^*T(H_{-c}) \), then \( g \) is expressible in the form

\[
g(z) = (T^*TG, K(\cdot, \bar{z}; -c))_{H_{-c}}
\]

\[
= (G, T^*TK(\cdot, \bar{z}; -c))_{H_{-c}}, \quad z \in \mathbb{C},
\]

for a member \( G \) in \( H_{-c} \) with \( T^*TG = g \). Following a general method in [8] or [9, p. 82], we shall give a characterization of the members in \( T^*T(H_{-c}) \).

From the representation (3.5) of \( T^*T \), we calculate the kernel form

\[
k(z, \bar{u}; c) = (T^*TK(\cdot, \bar{u}; -c), T^*TK(\cdot, \bar{z}; -c))_{H_{-c}}
\]

\[
= (TK(\cdot, \bar{u}; -c), TT^*TK(\cdot, \bar{z}; -c))_{L^2(\mu)}.
\]
Meanwhile, for all $z' \in \mathbb{C}$ we have
\[
[T^*TK(\cdot, \overline{z}; -c)](z') = (TK(\cdot, \overline{z}; -c), TK(\cdot, \overline{z'}; -c))_{L^2(\mu)}
\]
\[
= \frac{1}{(1 - |w|^4)\sqrt{\pi}} \exp \left\{ -\frac{|w|^4 z^2}{1 - |w|^4} \right\} \exp \left\{ -\frac{|w|^4 \overline{z'}^2}{1 - |w|^4} \right\} \cdot \int_{\mathbb{R}} \exp \left\{ -\frac{1 + |w|^4 \xi^2}{1 - |w|^4} + \frac{2|w|^2 (\overline{z} + z')}{1 - |w|^4} \xi \right\} d\xi
\]
\[
= \frac{1}{\sqrt{1 - |w|^8}} \exp \left\{ -\frac{|w|^8 z^2}{1 - |w|^8} \right\} \exp \left\{ -\frac{|w|^8 \overline{z'}^2}{1 - |w|^8} \right\} \exp \left\{ 2|w|^4 z' \overline{z} \frac{1}{1 - |w|^4} \right\}.
\]
Hence, the desired kernel form is given by
\[
k(z, \overline{u}; c) = \frac{1}{\sqrt{\pi(1 - |w|^4)(1 - |w|^8)}} \exp \left\{ -\frac{|w|^8 z^2}{1 - |w|^8} \right\} \exp \left\{ -\frac{|w|^8 \overline{u}^2}{1 - |w|^8} \right\} \cdot \int_{\mathbb{R}} \exp \left\{ -\frac{|w|^6 + |w|^4 + 1}{1 - |w|^8} \xi^2 + \left( \frac{2|w|^4 z}{1 - |w|^4} + \frac{2|w|^2 \overline{u}}{1 - |w|^4} \right) \xi \right\} d\xi
\]
\[
= \frac{1}{\sqrt{(1 - |w|^4)(|w|^8 + |w|^4 + 1)}} \exp \left\{ -\frac{|w|^4 z^2}{1 - |w|^4} \right\} \cdot \exp \left\{ \frac{2|w|^6 z \overline{u}}{(1 - |w|^4)(|w|^8 + |w|^4 + 1)} \right\}.
\]
Then the range $T^*T(H_{-c})$ coincides with the reproducing kernel Hilbert space $H_{k(c)}$ admitting the reproducing kernel $k(z, \overline{u}; c)$, and $T^*T$ is an isometry of $H_{-c}$ onto $H_{k(c)}$. Let $H$ be the reproducing kernel Hilbert space determined by the positive matrix $\exp[2|w|^6 z \overline{u}/((1 - |w|^4)(|w|^8 + |w|^4 + 1))]$. As in the proof of Theorem 1.2, we obtain the factorization
\[
g(z) = \frac{1}{\sqrt{(1 - |w|^4)(|w|^8 + |w|^4 + 1)}} \cdot \exp \left\{ -\frac{|w|^6 z^2}{1 - |w|^4} \right\} g_1(z)
\]
for an entire function $g_1 \in H$. So it gives the norm identity
\[
\|g\|^2_{H_{k(c)}} = \frac{1}{\sqrt{(1 - |w|^4)(|w|^8 + |w|^4 + 1)}} \|g_1\|^2_H,
\]
where
\[
\|g_1\|^2_H = \frac{2|w|^6}{\pi(1 - |w|^4)(|w|^8 + |w|^4 + 1)} \cdot \int_{\mathbb{C}} |g_1(z)|^2 \exp \left\{ -\frac{2|w|^6 |z|^2}{(1 - |w|^4)(|w|^8 + |w|^4 + 1)} \right\} dx dy.
\]
Note that $H$ is the totality of entire functions with finite norms (3.10). Hence, the space $H_{k(c)}$ consists of entire functions with finite norms

\[
\|g\|_{H_{k(c)}}^2 = \frac{2|w|^6}{\pi \sqrt{(1 - |w|^4)(|w|^8 + |w|^4 + 1)}} \cdot \int_{\mathbb{C}} |g(z)|^2 \exp \left[ -\frac{2|w|^2}{(1 - |w|^4)(|w|^8 + |w|^4 + 1)} \right. \\
\left. \cdot \{(1 - |w|^6)x^2 + (1 + |w|^6)y^2\} \right] dx \, dy.
\]

(3.11)

Since $T^*f$ is expressible in the form

\[
[T^*f](z) = \frac{1}{\sqrt{\pi (1 - |w|^4)}} \exp \left\{ -\frac{|w|^4 z^2}{1 - |w|^4} \right\} \int_{\mathbb{R}} f(\xi) \exp \left\{ \frac{(2|w|^2 z - \xi)\xi}{1 - |w|^4} \right\} d\xi,
\]

our claim has been proved.

When a $C^\infty$-function $f \in L^2(\mu)$ satisfies one of three items in Theorem 3.1, we shall give the representation of the analytic extension $\hat{f}$ in terms of $f$. By the isometry $T^*T$ of $H_{-c}$ onto $H_{k(c)}$, we have

**Theorem 3.2.** Let $c$, $w$, and $f$ be as in Theorem 3.1. If $f$ satisfies one of three items in Theorem 3.1 and $\hat{f}$ denotes the analytic extension of $f$ to $\mathbb{C}$, then $\hat{f}$ is represented by

\[
\hat{f}(z) = \frac{2|w|^6}{\pi^{3/2} (1 - |w|^4) \sqrt{(1 - |w|^4)(|w|^8 + |w|^4 + 1)}} \exp \left\{ -\frac{|w|^8 z^2}{1 - |w|^8} \right\} \\
\cdot \int_{\mathbb{C}} \left[ \int_{\mathbb{R}} f(\xi) \exp \left\{ \frac{(2|w|^2 z - \xi)\xi}{1 - |w|^4} \right\} d\xi \right. \\
\left. \cdot \exp \left[ -\frac{|w|^4}{1 - |w|^8} (Z^2 + |w|^4Z^2 - 2\overline{Z} z) \right. \right. \\
\left. \left. - \frac{2|w|^6}{1 - |w|^8} \{(1 - |w|^6)x^2 + (1 + |w|^6)y^2\} \right) \right] \frac{dX \, dY}{(1 - |w|^4)(|w|^8 + |w|^4 + 1)},
\]

(3.13)

where $Z = X + iY$.

**Proof.** Let $S$ be the adjoint operator of $T^*T$ from $H_{k(c)}$ to $H_{-c}$. Then, since $T^*T$ is an isometry of $H_{-c}$ onto $H_{k(c)}$, the operator $S$ is the inverse of $T^*T$. Hence, $T^*f = T^*TST^*f$ and $f = TST^*f$. Since

\[
\hat{f}(z) = [ST^*f](z) = (ST^*f, K(\cdot, \overline{z}; -c))_{H_{-c}} = (T^*f, T^*TK(\cdot, \overline{z}; -c))_{H_{k(c)}},
\]

from (3.7), (3.11), and (3.12) we obtain the desired representation (3.13).

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