

## CONVEX REAL PROJECTIVE STRUCTURES ON CLOSED SURFACES ARE CLOSED

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**ABSTRACT.** The deformation space  $\mathcal{C}(\Sigma)$  of convex  $\mathbb{RP}^2$ -structures on a closed surface  $\Sigma$  with  $\chi(\Sigma) < 0$  is closed in the space  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$  of equivalence classes of representations  $\pi_1(\Sigma) \rightarrow \text{SL}(3, \mathbb{R})$ . Using this fact, we prove Hitchin's conjecture that the contractible "Teichmüller component" (*Lie groups and Teichmüller space*, preprint) of  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$  precisely equals  $\mathcal{C}(\Sigma)$ .

Let  $\Sigma$  be a closed orientable surface of genus  $g > 1$  and  $\pi = \pi_1(\Sigma)$  its fundamental group. A convex  $\mathbb{RP}^2$ -structure on  $M$  is a representation (uniformization) of  $M$  as a quotient  $\Omega/\Gamma$  where  $\Omega \subset \mathbb{RP}^2$  is a convex domain and  $\Gamma \subset \text{SL}(3, \mathbb{R})$  is a discrete group of collineations of  $\mathbb{RP}^2$  acting properly and freely on  $\Omega$ . (See [5] for basic theory of such structures.) The space of projective equivalence classes of convex  $\mathbb{RP}^2$ -structures embeds as an open subset in the space of equivalence classes of representations  $\pi \rightarrow \text{SL}(3, \mathbb{R})$ . The purpose of this note is to show that this subset is also closed.

In [7], Hitchin shows that the space of equivalence classes of representations  $\pi \rightarrow \text{SL}(3, \mathbb{R})$  falls into three connected components: one component  $C_{-1}$  consisting of classes of representations for which the associated flat  $\mathbb{R}^3$ -bundle over  $\Sigma$  has nonzero second Stiefel-Whitney class; a component  $C_0$  containing the class of the trivial representation; a component  $C_1$  diffeomorphic to a cell of dimension  $16(g-1)$ , which he calls the "Teichmüller component." While  $C_{-1}$  can be distinguished from  $C_0$  and  $C_1$  by a topological invariant [3, 4], no characteristic invariant distinguishes representations in the Teichmüller component from those in  $C_0$ . The Teichmüller component is defined as follows. Using the Klein-Beltrami model of hyperbolic geometry, a hyperbolic structure on  $\Sigma$  is a special case of a convex  $\mathbb{RP}^2$ -structure  $\Omega/\Gamma$  where  $\Omega$  is the region bounded by a conic. In this case  $\Gamma$  is conjugate to a cocompact lattice in  $\text{SO}(2,1) \subset \text{SL}(3, \mathbb{R})$ . The space  $\mathfrak{T}(\Sigma)$  of hyperbolic structures ("Teichmüller

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space”) is a cell of dimension  $6(g - 1)$ , which is a connected component of  $\text{Hom}(\pi, \text{SO}(2, 1))/\text{SO}(2, 1)$ . Regarding hyperbolic structures on  $\Sigma$  as convex  $\mathbb{RP}^2$ -structures embeds the Teichmüller space  $\mathfrak{T}(\Sigma)$  inside  $\mathfrak{C}(\Sigma)$ . By [5], the space  $\mathfrak{C}(\Sigma)$  of convex  $\mathbb{RP}^2$ -structures on a compact surface  $\Sigma$  is shown to be diffeomorphic to a cell of dimension  $16(g - 1)$  and  $\mathfrak{T}(\Sigma)$  embeds  $C_1$  as the space of equivalence classes of embeddings of  $\pi$  as discrete subgroups of  $\text{SO}(2, 1) \subset \text{SL}(3, \mathbb{R})$ . Hitchin’s component  $C_1$  can thus be characterized as the component of  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$  containing equivalence classes of discrete embeddings  $\pi \rightarrow \text{SO}(2, 1)$ .

**Theorem A.** *Hitchin’s Teichmüller component  $C_1$  equals the deformation space  $\mathfrak{C}(\Sigma)$  of convex  $\mathbb{RP}^2$ -structures on  $\Sigma$ .*

In [5, 3.3] it is shown that the deformation space  $\mathfrak{C}(\Sigma)$  is an open subset of  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$  containing  $\mathfrak{T}(\Sigma)$  and hence an open subset of  $C_1$ . Let

$$\Pi: \text{Hom}(\pi, \text{SL}(3, \mathbb{R})) \rightarrow \text{Hom}(\pi, \text{SL}(3, \mathbb{R}))/\text{SL}(3, \mathbb{R})$$

denote the quotient map. Also by [5, 3.2], every representation in  $\Pi^{-1}(\mathfrak{C}(\Sigma))$  has image Zariski-dense in either a conjugate of  $\text{SO}(2, 1)$  or  $\text{SL}(3, \mathbb{R})$  itself, and hence by [5, 1.12]  $\text{SL}(3, \mathbb{R})$  acts properly and freely on  $\Pi^{-1}(\mathfrak{C}(\Sigma))$ . In particular, the restriction

$$\Pi: \Pi^{-1}(\mathfrak{C}(\Sigma)) \rightarrow \mathfrak{C}(\Sigma)$$

is a locally trivial principal  $\text{SL}(3, \mathbb{R})$ -bundle. It follows that  $\Pi^{-1}(\mathfrak{C}(\Sigma))$  is an open subset of  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$ . Thus Theorem A is a corollary of

**Theorem B.**  *$\Pi^{-1}(\mathfrak{C}(\Sigma))$  is a closed subset of  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$ .*

The rest of the paper is devoted to the proof of Theorem B. Arguments similar to the proof are given at the end of the first chapter of [1] and an analogous statement when  $\Sigma$  is a pair-of-pants is proved in [5, §§4.4 and 4.5] (where it is used in the proof of the main theorem). We feel there is a more comprehensive result for compact surfaces with boundary, with a geometric proof.

Assume that  $\phi_n$  is a sequence of representations in  $\text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$  which converges to  $\phi \in \text{Hom}(\pi, \text{SL}(3, \mathbb{R}))$  and that each  $\Pi(\phi_n) \in \mathfrak{C}(\Sigma)$ . Thus for each  $n$ , there exists a convex domain  $\Omega_n \subset \mathbb{RP}^2$  such that  $\phi_n: \pi \rightarrow \text{SL}(3, \mathbb{R})$  embeds  $\pi$  onto a discrete group  $\Gamma_n$  acting properly and freely on  $\Omega_n$ . Furthermore, as discussed in [5, 3.2(1)], each  $\Omega_n$  is strictly convex and has the property that the closure  $\overline{\Omega_n}$  is a compact subset of an affine patch (the complement of a projective line) in  $\mathbb{RP}^2$ .

We identify the universal covering of  $\mathbb{RP}^2$  with the 2-sphere  $S^2$  of oriented directions in  $\mathbb{R}^3$ . Denote by  $p: S^2 \rightarrow \mathbb{RP}^2$  the covering projection. The group

$$\text{SL}_{\pm}(3, \mathbb{R}) = \{A \in \text{GL}(3, \mathbb{R}) \mid \det(A) = \pm 1\}$$

acts on  $S^2$  covering the action of  $\text{SL}(3, \mathbb{R})$  on  $\mathbb{RP}^2 = S^2/\{\pm 1\}$ . A choice of positive definite inner product on  $\mathbb{R}^3$  realizes  $S^2$  as the unit sphere in  $\mathbb{R}^3$ , and  $d: S^2 \times S^2 \rightarrow \mathbb{R}$  denotes the distance function corresponding to the induced Riemannian metric. The geodesics in  $S^2$  are arcs of great circles. If  $\Omega \subset \mathbb{RP}^2$  has the property that there exists an affine patch  $A \subset \mathbb{RP}^2$  such that  $\Omega \subset A$  is convex (with respect to the affine geometry on  $A$ ), then we say that  $\Omega$

is *properly convex*. In that case each component of  $p^{-1}(\Omega)$  is convex in the corresponding elliptic geometry of  $S^2$  and there exists a sharp convex cone in  $\mathbb{R}^3$  whose projectivization equals  $\Omega$ . We shall also refer to a component of  $p^{-1}(\Omega)$  as properly convex. (A *sharp convex cone* in an affine space  $E$  is an open convex domain  $\Omega \subset E$  invariant under positive homotheties and containing no complete affine line.)

Since an affine patch is contractible,  $p^{-1}(\Omega_n)$  consists of two components each of which maps diffeomorphically to  $\Omega_n$ . Choose one of the components  $\Omega'_n \subset S^2$  for each  $n$ . Furthermore  $\phi_n$  defines an effective proper action of the discrete group  $\pi$  on  $\Omega'_n$  whose quotient is a convex  $\mathbb{RP}^2$ -surface homeomorphic to  $\Sigma$ . Moreover, since  $\pi$  is not virtually nilpotent and  $\phi_n$  is a discrete embedding for each  $n$ , the limiting representation  $\phi = \lim_{n \rightarrow \infty} \phi_n$  is also a discrete embedding (see, e.g., [6, Lemma 1.1]). In particular, the image  $\Gamma$  of  $\phi$  is torsionfree and not virtually abelian.

Since the space of compact subsets of  $S^2$  is compact in the Hausdorff topology, we may (after extracting a subsequence) assume that the sequence  $\overline{\Omega'_n}$  converges (in the Hausdorff topology) to a compact subset  $K \subset S^2$ .

**Lemma 1.**  *$K$  is invariant under the image  $\Gamma = \phi(\pi)$ .*

*Proof.* Suppose that  $k \in K$  and  $g \in \pi$ . We show that  $\phi(g)k \in K$ . Let  $\varepsilon > 0$ . Now  $\phi_n(g)$  converges uniformly to  $\phi(g)$  on  $S^2$ ; thus there exists  $N_1 = N_1(\varepsilon)$  such that

$$d(\phi_n(g)x, \phi(g)x) < \varepsilon/2$$

for  $n > N_1$ . Indeed the family  $\phi_n(g)$  is uniformly Lipschitz for sufficiently large  $n$ —let  $C$  be a Lipschitz constant, i.e.,

$$d(\phi_n(g)x, \phi_n(g)y) \leq Cd(x, y)$$

for all  $x, y \in S^2$  and  $n$  sufficiently large, say  $n > N_2$ . Since  $K$  is the Hausdorff limit of  $\overline{\Omega'_n}$ , there exist  $w_n \in \overline{\Omega'_n}$  such that  $w_n \rightarrow k$ . Thus there exists  $N_3 = N_3(\varepsilon)$  such that  $d(k, w_n) < \varepsilon/(2C)$  for  $n > N_3$ . Putting these inequalities together, we obtain

$$\begin{aligned} d(\phi(g)k, \phi_n(g)w_n) &\leq d(\phi(g)k, \phi_n(g)k) + d(\phi_n(g)k, \phi_n(g)w_n) \\ &< \varepsilon/2 + C\varepsilon/(2C) = \varepsilon \end{aligned}$$

for  $n > \max(N_1, N_2, N_3)$ . It follows that  $\phi(g)k$  is the limit of  $\phi_n(g)w_n \in \overline{\Omega'_n}$ . Since the Hausdorff limit of  $\overline{\Omega'_n}$  equals  $K$ , it follows that  $\phi(g)k \in K$ , as claimed.  $\square$

Furthermore each  $\overline{\Omega'_n}$  is convex in  $S^2$ . Since convex sets are closed in the Hausdorff topology, it follows that  $K$  is also convex. (See [2] for more details.)

There are the following possibilities for  $K$  (compare Choi [2]):

- (1)  $K$  is properly convex with nonempty interior.
- (2)  $K$  consists of a single point.
- (3)  $K$  consists of a line segment.
- (4)  $K$  is a great disk (i.e., a closed hemisphere).

We show that only case (1) can arise. The following lemma (whose proof we defer) is used to rule out the last three cases.

**Lemma 2.** *Suppose that  $F$  is a nonabelian free group and  $h: F \rightarrow \text{SL}(2, \mathbb{R})$  is a homomorphism which embeds  $F$  onto a discrete subgroup of  $\text{SL}(2, \mathbb{R})$ . Then there exists  $f \in F$  such that  $h(f)$  has negative trace.*

In cases (2)–(4), there is either a projective line or a point in  $\mathbb{RP}^2$  which is invariant under the stabilizer  $G$  of  $K$ . In each of these cases  $G$  is conjugate to one of the subgroups of  $\text{SL}(3, \mathbb{R})$  consisting of matrices

$$\begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \quad \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

In both cases, there is a homomorphism  $\rho: G \rightarrow \text{SL}(2, \mathbb{R})$  such that if  $g \in [G, G]$  then

$$\text{tr}(g) = 1 + \text{tr}(\rho(g)).$$

(We take  $g$  to lie in the commutator subgroup so as to assume that the (1,1)-matrix entry and the determinant of the  $(2 \times 2)$ -block are both 1.)

We suppose that  $\phi_n$  is a sequence as above converging to  $\phi$ . Since  $\phi$  is a discrete embedding, apply Lemma 2 to the restriction  $h$  of  $\rho \circ \phi$  to  $F = [\pi, \pi]$ . We deduce that there exists  $\gamma \in \pi$  such that  $\text{tr}(\phi(\gamma)) < 1$ . However, as discussed in [5, 3.2(3)], every  $1 \neq \gamma \in \pi$  has the property that  $\phi_n(\gamma) \in \text{SL}(3, \mathbb{R})$  has positive eigenvalues; in particular,  $\text{tr} \phi_n(\gamma) > 3$ . Since  $\phi_n \rightarrow \phi$ , it follows that  $\text{tr} \phi(\gamma) \geq 3$ , a contradiction.

Thus only case (1) is possible:  $K$  is properly convex with interior  $\Omega$ . Then  $\Gamma$  acts isometrically with respect to the Hilbert metric on  $\Omega$ . Since  $\Gamma$  is discrete, torsionfree, and acts properly on  $\Omega$ , the quotient  $\Omega/\Gamma$  is a closed surface. Since  $\Gamma$  is not virtually abelian,  $\Omega$  is not a triangular region and by [8] (see also [5, 3.2]) it follows that  $\Omega/\Gamma$  is a convex  $\mathbb{RP}^2$ -manifold homeomorphic to  $\Sigma$ . This concludes the proof of Theorem B, assuming Lemma 2.

*Proof of Lemma 2.* By passing to a subgroup of  $F$  we may assume that the quotient of the hyperbolic plane by the image of  $h(F)$  under the quotient homomorphism  $\text{SL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$  is a complete surface which is homeomorphic to a pair-of-pants  $P$  (a sphere minus three discs). Let  $f_1, f_2, f_3$  be elements of  $\pi_1(P) \subset F$  corresponding to the three components of  $\partial P$ . We choose elements  $\widetilde{h}(f_j) \in \text{SL}(2, \mathbb{R})$  ( $j = 1, 2, 3$ ) so that  $\text{tr}(\widetilde{h}(f_j)) > 2$  (equivalently,  $\widetilde{h}(f_j)$  lies in a hyperbolic one-parameter subgroup of  $\text{SL}(2, \mathbb{R})$ ). Now  $f_1 f_2 f_3 = 1$  in  $F$  but

$$\widetilde{h}(f_1)\widetilde{h}(f_2)\widetilde{h}(f_3) = (-1)^{\chi(P)} = -1$$

(since the relative Euler class of the representation equals  $-1$ ; compare the discussion in [4, §4]). Since each  $h(f_i)$  is hyperbolic, an odd number of  $f_i$  must satisfy  $\text{tr}(h(f_i)) < 2$ ; in particular, at least one  $f \in F$  satisfies  $\text{tr}(h(f)) < 0$ .  $\square$

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