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COUNTEREXAMPLES FOR BROWN-PEDERSEN'S CONJECTURE IN "C^*-ALGEBRAS OF REAL RANK ZERO"

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ABSTRACT. We give nonseparable C^*-algebras which are counterexamples for Brown-Pedersen's conjecture in "C^*-algebras of real rank zero".

1. INTRODUCTION

Let A be a unital C^*-algebra and A_{sa} be the set of all selfadjoint elements in A. For nonnegative integer n, consider the set of left n-tuples

$$L_n(A) = \left\{ (a_0, a_1, \dots, a_n) \in A_{\mathrm{sa}}^{n+1};\ \sum_{k=0}^{n} Aa_k = A \right\}.$$

The real rank of A, denoted $\mathrm{RR}(A)$, is the least integer n such that

$$L_n(A) \text{ is dense in } A_{\mathrm{sa}}^{n+1}.$$

If A is nonunital, its real rank is defined by $\mathrm{RR}(\widetilde{A})$, where $\widetilde{A} = A \oplus \mathbb{C}$.

In [1] Brown and Pedersen defined and studied this concept and especially characterized C^*-algebras of real rank zero. The following three conjectures are given:

(1) $\mathrm{RR}(M(A)) = 0$ for every AF-algebra A,
(2) $\mathrm{RR}(M(A)) = 0$ for every C^*-algebra A for which $\mathrm{RR}(A) = 0$ and $K_1(A) = 0$, and
(3) if A is a C^*-algebra with $\mathrm{RR}(A) = 0$, then $\mathrm{RR}(M(A)/A) = 0$,

where $M(A)$ denotes the multiplier C^*-algebra of A and AF-algebras mean approximately finite-dimensional C^*-algebras. Recently, conjecture (1) has been proved by Lin [4] in the σ-unital case.

In this note, we present counterexamples for these three conjectures, although those C^*-algebras are non-σ-unital.

We refer the reader to [1] for results about real ranks.

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2. Counterexamples

First we introduce a normal space Y with the covering dimension $(= \dim Y)$ $\neq 0$, which is Dowker's example [2, 6.2.20].

Let Q denote the set of all rational numbers in the interval $I = [0, 1]$. By taking

$$x \sim y \quad \text{if and only if} \quad |x - y| \in Q$$

we define an equivalence relation on I. Then the family of all equivalence classes has cardinality $\aleph_1$. Choose a subfamily of cardinality $\aleph_1$ which does not contain the equivalence class Q, and let us arrange its members into a transfinite sequence $Q_0, Q_1, \dots, Q_a, \dots, a < \omega_1$, where a is an ordinal number and ω_1 is the smallest uncountable ordinal number of cardinality $\aleph_1$.

For every $a < \omega_1$ the set $S_a = I \setminus \bigcup_{\gamma \geq a} Q_\gamma$ is zero-dimensional, that is, a nonempty T_1-space, and has a base consisting of open and closed sets. Let X be the space of all ordinal numbers $\leq \omega_1$. Consider on X the topology generated by the base consisting of all segments $(y, x] = \{z \in X;\ y < z \leq x\}$ $(y < x \leq \omega_1)$ and the one-point $\{0\}$, where 0 is the order type of the empty set. We know that X is a compact Hausdorff space. For every $a < \omega_1$, $X_a = \{\gamma \in X; \gamma \leq a\}$ is open and closed in X; hence, the Cartesian product $X_a \times S_a$ is a zero-dimensional metrizable space. Let $Y_a = \bigcup_{\gamma \leq a}(\{\gamma\} \times S_\gamma)$ and $Y = \bigcup_{a<\omega_1} Y_a$. Since $Y_a = Y \cap (X_a \times I)$, it is open and closed in Y and a zero-dimensional metrizable space; hence, $\dim Y_a = 0$. But we know $\dim Y \gneq 0$ due to Dowker [2, 6.2.20].

Since Y is not locally compact, we modify Y to a locally compact space by Isbell's result [3, Chapter VI.14].

Let W be the set $X - \{\omega_1\}$, and let $p\colon Y \to W$ be the first coordinate projection. Let βY be the Stone-Čech compactification of Y. Then we have an extension map $\beta p\colon \beta Y \to \beta W$, and let Z be $(\beta p)^{-1}(W)$. It is obvious that Z is locally compact. Since $Y \subset Z \subset \beta Y$, $\beta Z = \beta Y$ and $\dim \beta Z = \dim \beta Y \gneq 0$. Let $Z_a = Z \cap (X_a \times I)$, then $Z = \bigcup_{a<\omega_1} Z_a$. Since the small inductive dimension of Z $(= \operatorname{ind} Z) = 0$, $\operatorname{ind} Z_a = 0$ by [2, Theorem 7.1.1] and $\dim Z_a = 0$ by [2, Theorem 7.1.11].

Let $C_0(Z)$ be the commutative C^*-algebra of all complex continuous functions of Z vanishing at infinity. By the previous construction for Z, $C_0(Z)$ is an inductive limit of $C_0(Z_a)$. As $\dim Z_a = 0$, $C_0(Z_a)$ is AF, and hence $C_0(Z)$ is AF.

We get counterexamples for the previous three conjectures in the nonseparable case.

Theorem. *Let n be any positive integer ≥ 1. Consider the following C^*-exact sequence:*

$$\begin{aligned} 0 \to C_0(Z) \otimes M_n(\mathbb{C}) &\to M(C_0(Z) \otimes M_n(\mathbb{C})) \\ &\to M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C}) \to 0. \end{aligned}$$

Then $\mathrm{RR}(C_0(Z) \otimes M_n(\mathbb{C})) = 0$, $\mathrm{RR}(M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C})) \neq 0$, *and* $\mathrm{RR}(M(C_0(Z) \otimes M_n(\mathbb{C}))) \neq 0$.

Proof. Since $C_0(Z)$ is AF, so is $C_0(Z) \otimes M_n(\mathbb{C})$ for any integer $n \geq 1$, and $\mathrm{RR}(C_0(Z) \otimes M_n(\mathbb{C})) = 0$ by [1, Proposition 3.1].

It is obvious that $M(C_0(Z) \otimes M_n(\mathbb{C})) = C(\beta Z) \otimes M_n(\mathbb{C})$. Since $\dim Z = \dim \beta Z \gneq 0$ (see [2, Theorem 7.1.17]), $\mathrm{RR}(C(\beta Z)) = \dim Z \gneq 0$ from [1, Proposition 1.1]. Moreover, we know $\mathrm{RR}(C(\beta Z) \otimes M_n(\mathbb{C})) \gneq 0$ from [1, Corollary 2.8].

Suppose that $\mathrm{RR}(C(\beta Z \setminus Z)) = 0$. By [1, Proposition 1.1] $\dim \beta Z \setminus Z = 0$, and $C(\beta Z \setminus Z)$ is AF. Since every projection in $C(\beta Z \setminus Z)$ lifts to a projection in $C(\beta Z)$, we know $\mathrm{RR}(C(\beta Z)) = 0$ from [1, Theorem 3.14], which is a contradiction. Hence, $\mathrm{RR}(C(\beta Z \setminus Z)) \gneq 0$. As $M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C}) = C(\beta Z \setminus Z) \otimes M_n(\mathbb{C})$, we know

$$\mathrm{RR}(M(C_0(Z) \otimes M_n(\mathbb{C}))/C_0(Z) \otimes M_n(\mathbb{C})) \gneq 0$$

as in the above argument. □

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