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INDEPENDENT EVENTS AND INDEPENDENT EXPERIMENTS

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ABSTRACT. It is shown that independent events in a probability space with equally likely outcomes are isomorphic to events coming from independent experiments each with equally likely outcomes.

INTRODUCTION

Let Ω be a finite set with uniform probability distribution P. Denote the cardinality of Ω by $|\Omega| = \prod_{j=1}^{n} p_j$, where the p_j's are prime numbers greater than one, which are not necessarily distinct. Eisenberg and Ghosh [1] show that there does not exist a collection of more than n nontrivial mutually independent events in Ω, where n is the number of prime factors of $|\Omega|$. An event is called *trivial* if it has probability 0 or 1. The proof is complicated by the fact that it uses the number of times that each distinct prime number occurs in the factorization of $|\Omega|$. In this article we give a more unified proof of the result of Eisenberg and Ghosh.

Next consider the product space $\Omega_1 \times \cdots \times \Omega_n$ with the uniform probability distribution where $|\Omega_j| = p_j$. The Ω_j's can be interpreted as spaces of outcomes of independent experiments, each with equally likely outcomes. If A_j is a nontrivial cylinder set based on Ω_j for $j = 1, \ldots, n$, then $A_1, \ldots, A_n$ form a collection of n nontrivial mutually independent events in $\Omega_1 \times \cdots \times \Omega_n$. These are the most fundamental types of independent events since they are events related to independent experiments. There is a 1-1 map from this product space to Ω, and under this isomorphism the images of the A_j's are nontrivial mutually independent events in Ω. In this article we show that any set of nontrivial mutually independent events in Ω must be of this form; that is, mutually independent events in Ω must be isomorphic to events from independent experiments with equally likely outcomes.

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1. Independent events in a space of N equally likely outcomes

The main results are corollaries to the following lemma. The notation of the lemma is used throughout this section.

Lemma. *Let Ω be a probability space with N equally likely outcomes. Assume that $A_1, \dots, A_m$ are nontrivial mutually independent events in Ω. Let $\mathfrak{F}_0 = \{\phi, \Omega\}$ and let $\mathfrak{F}_j$ be the field generated by $\{A_1, \dots, A_j\}$ for $j = 1, \dots, m$. Let N_j be the greatest common divisor of the cardinalities of the sets in $\mathfrak{F}_j$. Then*

(1) *$N_0 = N$ and N_j is a proper divisor of N_{j-1} for $j = 1, \dots, m$ and*
(2) *$P(A_j) = h_j N_j / N_{j-1}$ for an integer h_j with $1 \le h_j < N_{j-1}/N_j$.*

Proof. Since $\mathfrak{F}_{j-1} \subset \mathfrak{F}_j$, it follows that N_j divides the cardinality of every set in $\mathfrak{F}_{j-1}$, and hence N_j divides N_{j-1}. We must show that it is a proper divisor, i.e., $N_j < N_{j-1}$.

From the definition of N_{j-1}, it follows that there exists a representation $N_{j-1} = \sum_\alpha k_\alpha |B_\alpha|$, where the k_α's are integers and the B_α's are sets in $\mathfrak{F}_{j-1}$. Form the sum $S = \sum k_\alpha |B_\alpha A_j|$. Since A_j is independent of $\mathfrak{F}_{j-1}$, it follows that $|B_\alpha A_j| = P(B_\alpha A_j)N = P(B_\alpha)P(A_j)N = |B_\alpha| P(A_j)$. Thus $S = \sum k_\alpha |B_\alpha| P(A_j) = N_{j-1} P(A_j)$. Since S is a sum of integral multiples of the cardinalities of sets in $\mathfrak{F}_j$, it must be divisible by N_j. Thus $N_{j-1}P(A_j) = h_j N_j$ for some integer h_j. Since $0 < P(A_j) < 1$, it follows that $1 \le h_j < N_{j-1}/N_j$. Hence $N_j = P(A_j) N_{j-1}/h_j < N_{j-1}$. This proves (1).

Furthermore, $P(A_j) = h_j N_j / N_{j-1}$, which gives (2). □

Theorem 1. *If $A_1, \dots, A_m$ are nontrivial mutually independent events in Ω then $m \le n$, where n is the number of prime factors of Ω.*

Proof. Suppose $m > n$. From the lemma, N_1 is a proper divisor of N. Hence N_1 has at most $n-1$ prime factors. Continuing in this way, N_m has at most $n-m$ prime factors. Thus $N_n = 1$, which has no proper divisors. But the lemma implies that N_{n+1} must be a proper divisor of N_n. This is a contradiction. □

Theorem 2. *If $A_1, \dots, A_m$ are nontrivial mutually independent events in Ω, then there exists an isomorphism from Ω to $\Omega' = \Omega_1 \times \cdots \times \Omega_m \times \Omega_{m+1}$, where $|\Omega_j| = N_{j-1}/N_j$ for $j = 1, \dots, m$ and $|\Omega_{m+1}| = N_m$, such that A_j gets mapped to a cylinder set based in Ω_j for $j = 1, \dots, m$.*

Proof. According to the lemma, $P(A_j) = h_j N_j / N_{j-1}$ with $1 \le h_j < N_{j-1}/N_j$. Choose a set of h_j points in Ω_j and let A'_j be the cylinder set based on this set. Then if all points in Ω' are equally likely, we have that $P(A_j) = P(A'_j)$. Due to the independence of $A_1, \dots, A_m$ in Ω and $A'_1, \dots, A'_m$ in Ω', we have that each atom in the field generated by $A_1, \dots, A_m$ has the same probability as the corresponding atom in the field generated by $A'_1, \dots, A'_m$. Since $|\Omega| = |\Omega'|$, these atoms must have the same number of points as well. By choosing arbitrary 1-1 maps between points in the corresponding atoms, we get the desired isomorphism. □

Corollary. *If $A_1, \dots, A_n$ are nontrivial mutually independent events in Ω, where n is the number of prime factors $|\Omega|$, then one can choose $\Omega' = \Omega_1 \times \cdots \times \Omega_n$ in Theorem 2.*

Proof. As in the proof of Theorem 1 we have that $N_n = 1$ so that the factor Ω_{n+1} is not needed in order that $|\Omega| = |\Omega'|$. The proof of Theorem 2 then applies to complete the proof. □

Example. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{2, 4, 6\}$, and $A_2 = \{1, 2, 3, 4\}$. Then A_1 and A_2 are independent events. $N = N_0 = 6$, $N_1 = 3$, and $N_2 = 1$. $N_0/N_1 = 2$ and $N_1/N_2 = 3$. Let $\Omega_1 = \{1, 2\}$ and $\Omega_2 = \{1, 2, 3\}$. We may then choose $A_1' = \{(1, 1), (1, 2), (1, 3)\}$ and $A_2' = \{(1, 1), (2, 1), (1, 2), (2, 2)\}$ so that A_1' represents the event "1 on experiment 1" and A_2' represents the event "1 or 2 on experiment 2." The isomorphism is then any 1-1 map that takes $A_1A_2 = \{2, 4\}$ to $A_1'A_2' = \{(1, 1), (1, 2)\}$, that takes $A_1A_2^c = \{6\}$ to $A_1'A_2'^c = \{(1, 3)\}$, that takes $A_1^cA_2 = \{1, 3\}$ to $A_1'^cA_2' = \{(2, 1), (2, 2)\}$, and takes $A_1^cA_2^c = \{5\}$ to $A_1'^cA_2'^c = \{(2, 3)\}$. One way to achieve this is $1 \to (2, 1)$, $2 \to (1, 1)$, $3 \to (2, 2)$, $4 \to (1, 2)$, $5 \to (2, 3)$, and $6 \to (1, 3)$.

In general these conclusions do not apply to pairwise independent events. For example, Eisenberg and Ghosh show that if $|\Omega| = k^2$ for any $k \geq 2$, then there exists a collection of $k + 1$ nontrivial pairwise independent events in Ω. For example, if $|\Omega| = 25$ then there is a collection of six nontrivial pairwise independent events. On the other hand, there can be no more than two nontrivial mutually independent events in this case. Theorem 1 does extend to pairwise independent events, however, if the prime factors of $|\Omega|$ are all distinct. This result is also proved in the Eisenberg and Ghosh article.

References

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