SCHATTEN IDEAL BEHAVIOR
OF A GENERALIZED HARDY OPERATOR

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Abstract. We show the $S^p$-criteria and the cut-off result for an operator closely related to the classical Hardy inequality.

INTRODUCTION

For a function $g$ defined on $(0, \infty)$ we consider the operator $A_g$ acting on $L^2(0, \infty)$, which is given by the formula

$$A_g h(y) = \frac{g(y)}{y} \int_0^y h(x) \, dx.$$ 

We prove the $S^p$-criteria for $A_g$, $1 < p < \infty$, and the cut-off result at $p = 1$ which states that $A_g$ cannot be trace class. This cut-off for $A_g$ is related to the result of Rochberg in [R2]. In a more general sense it resembles the critical index behavior of commutators of singular integral operators found by Janson and Wolff in [JW] and the behavior of Hankel operators on Bergman spaces studied by Arazy, Fisher, and Peetre in [AFP]. The case $p = \infty$ is also contained in the paper [BK] by Bloom and Kerman. After this work was done the author was informed that closely related results had been obtained earlier by Birman and Solomyak in [BS].

The boundedness of $A_g$ on $L^2(0, \infty)$ for $g \equiv 1$ becomes the classical Hardy inequality. The operator $A_g$ is essentially a square root of a Calderon-Toeplitz operator $T_\mu$ defined in the context of the Haar wavelet, i.e.,

$$A_g^* A_g = P T_\mu P,$$

where $P : L^2(R) \to L^2(0, \infty)$ is the orthogonal projection, and also $T_\mu$ may be expressed in terms of $A_g^* A_g$. The symbol of $T_\mu$ is the measure $\mu$ defined on the upper half plane by the formula

$$\int_{-\infty}^{\infty} \int_0^{\infty} F(u, s) \, d\mu(u, s) = \int_0^{\infty} F(0, s) |g(s)|^2 \frac{ds}{s}.$$
The background for Calderón-Toeplitz operators is presented in [R1, N]. The result in [N] for Calderón-Toeplitz operators shows that for a positive measure\( \mu \) defined on the upper half plane and \( p \geq 1 \),
\[
T_\mu \in S^p \quad \text{if and only if} \quad \int_0^\infty \int_{-\infty}^\infty |\mu(D(u, s))|^p \, du \, ds < \infty,
\]
where \( D(u, s) \) is a hyperbolic disk with fixed radius centered at \((u, s)\).

1. Preliminaries

For a compact operator \( T \) acting on a Hilbert space the \( n \)th singular value of \( T \) is defined as
\[
s_n(T) = \inf\{\|T - S\| : \text{rank}(S) \leq n\}.
\]
The Schatten ideal \( S^p \), \( 1 \leq p < \infty \), is defined as the set of those compact operators \( T \) such that
\[
\|T\|_{S^p} = \left( \sum_n (s_n(T))^p \right)^{1/p} < \infty.
\]
In addition \( S^\infty \) stands for the algebra of bounded operators. We refer to [GK, Mc, Si, Z] for more information about Schatten ideals.

For \( g \) a measurable function defined on \((0, \infty)\)
\[
\|g\|_{p(L^2)} = \left( \sum_k \left( \int_{2^k}^{2^{k+1}} |g(x)|^2 \frac{dx}{x} \right)^{p/2} \right)^{1/p},
\]
and \( p(L^2) \) consists of those \( g \) for which \( \|g\|_{p(L^2)} \) is finite.

A small letter \( c \) denotes a constant which changes its value from place to place. A symbol \( \chi_A \) stands for the characteristic function of a set \( A \).

**Proposition 1.1** (see [Mc]). Let \( 0 < p \leq 1 \) and let \( T \) be a compact positive operator on a Hilbert space \( H \). If for some orthonormal basis \( \{\psi_i\} \) of \( H \)
\[
\sum_i |\langle T\psi_i, \psi_i \rangle|^p < \infty,
\]
then \( T \in S^p \) and
\[
\|T\|_{S^p} \leq \sum_i |\langle T\psi_i, \psi_i \rangle|^p.
\]

**Proposition 1.2.** Let \( 1 \leq p \leq \infty \) and let \( \{f_k\}, \{g_k\} \) be orthonormal sets in a Hilbert space \( H \). If \( T \in S^p \), then
\[
\left( \sum_k |\langle Tf_k, g_k \rangle|^p \right)^{1/p} \leq \|T\|_{S^p}.
\]
If \( T \) is compact, then
\[
\lim_{|k| \to \infty} \langle Tf_k, g_k \rangle = 0.
\]

**Proof.** The first part follows by a standard duality argument, and the second by an approximation argument.
Proposition 1.3. Let $1 \leq p_0, p_1 \leq \infty$, $0 \leq \theta \leq 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. If $T$ is a bounded operator from $l^{p_0}(L^2)$ into $S^{p_0}$ and from $l^{p_1}(L^2)$ into $S^{p_1}$, then it is also bounded from $l^p(L^2)$ into $S^p$ and
\[ ||T||_{l^p(L^2) \to S^p} \leq \|T\|_{l^{p_0}(L^2) \to S^{p_0}}^{1-\theta} \|T\|_{l^{p_1}(L^2) \to S^{p_1}}^\theta.\]

Proof. The proof follows from standard interpolation results; in particular, Theorem 5.1.2 in [BL] and Theorem 4.2.7 in [Z].

2. The main result

Theorem 2.1. Let $1 < p \leq \infty$ and let $g$ be a locally square integrable function.

1. The operator $A_g$ is in the Schatten ideal $S^p$ if and only if $g$ belongs to $l^p(L^2)$; moreover, the corresponding norms are equivalent. The operator $A_g$ is compact if and only if
\[ \lim_{|k| \to \infty} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} = 0.\]

2. If $A_g \in S^1$, then $g \equiv 0$.

Condition (2) states that the family of operators $\{A_g\}$ is "cut off" at $p = 1$.

Proof. We prove that the map $A$, $A(g) = A_g$, is bounded from $l^\infty(L^2)$ into $S^\infty$ and from $l^p(L^2)$ into $S^p$, $1 < p \leq 2$. It follows by Proposition 1.3 that it is bounded from $l^p(L^2)$ into $S^p$ for all $1 < p < \infty$.

We first consider $p = \infty$. We observe that
\[ \int_0^\infty \frac{g(y)}{y} \int_0^y h(x) \, dx \, dy = \sum_k \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} \left( \int_0^y |h(x)| \, dx \right)^2 \leq \sup_k \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y} \, dy \sum_k \frac{1}{2^k} \left( \int_0^{2^{k+1}} |h(x)| \, dx \right)^2 \]
and that
\[ \left( \sum_k \frac{1}{2^k} \left( \int_0^{2^{k+1}} |h(x)| \, dx \right)^2 \right)^{1/2} \leq c \left( \int_0^\infty \left( \frac{1}{y} \int_0^y |h(x)| \, dx \right)^2 \, dy \right)^{1/2} \leq c\|h\|, \]
where the last inequality is justified by the classical Hardy inequality. The above computation shows that $A$ is bounded from $l^\infty(L^2)$ into $S^\infty$.

Now let $1 < p < 2$. It is clear that $A_g^* A_g$ is given by the kernel
\[ K(x, z) = \int_0^\infty \frac{|g(y)|^2}{y^2} \chi_{[0, y]}(x) \chi_{[0, y]}(z) \, dy.\]

Let
\[ K_k(z, x) = \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} \chi_{[0, y]}(x) \chi_{[0, y]}(z) \, dy.\]
The operator given by $K_k$ acts on $L^2(0, 2^{k+1})$, and

$$e_k(x) = 2^{-(k+1)/2} e^{2\pi i z^{-(k+1)}}_{nx}$$

forms an orthonormal basis on that space. A direct computation gives

$$|(K_k e_k^n, e_k^n)| \leq \frac{c}{n^2} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y}.$$  

We obtain by Proposition 1.1

$$||K_k||_{S^p}^{p/2} \leq \sum_n |(K_k e_k^n, e_k^n)|^{p/2} \leq c \left( \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} \right)^{p/2},$$

and this provides the estimate

$$||A_g||_{S^p}^p = ||K||_{S^p}^p \leq \sum_k ||K_k||_{S^p}^{p/2} \leq c ||g||_{S^p}^p.$$  

Thus $A$ is bounded from $l^p(L^2)$ into $S^p$, $1 < p < 2$.

It is easy to observe at this point that if

$$\lim_{|k| \to \infty} \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} = 0,$$

then $A_g$ is compact.

Now for the converse estimate we assume that $A_g \in S^p$, and we take

$$f_k(x) = 2^{-(k-1)/2} \chi_{[2^{k-1}, 2^k]}(x),$$

$$g_k(y) = \frac{g(y)}{y} \chi_{[2^k, 2^{k+1}]}(y) \left( \int_{2^k}^{2^{k+1}} \frac{|g(y)|^2}{y^2} \frac{dy}{y} \right)^{-1/2}.$$  

It follows by Proposition 1.2 that

$$\sum_k |(A_g f_k, g_k)|^p \leq ||A_g||_{S^p}^p.$$  

A direct computation gives

$$\langle A_g f_k, g_k \rangle \approx \left( \int_{2^k}^{2^{k+1}} |g(y)|^2 \frac{dy}{y} \right)^{1/2},$$

and this proves the converse estimate.

Now we discuss the cut-off. Assume that for some $0 < a < b < \infty$

$$0 < \int_a^b |g(y)|^2 \frac{dy}{y} < \infty$$

and that $A_g \in S^1$. The kernel

$$B_g(y, x) = \chi_{(a, b)}(y) \frac{g(y)}{y} \chi_{[0, y]}(x) \chi_{(0, b)}(x)$$

defines an operator in $S^1(0, b)$. Let $e_n(x) = e^{2\pi ibnx}$. It is easy to check that

$$\langle B_g e_n, e_n \rangle = c(\hat{g}_0(0) - \hat{g}_0(n))/n,$$
where \( g_0(y) = \frac{g(y)}{y} \chi(a, b)(y) \). Changing \( a, b \) if necessary we may arrange that
\[
\hat{g}_0(0) = \int_a^b \frac{g(y)}{y} \, dy \neq 0.
\]
By the Riemann-Lebesgue lemma \( \hat{g}_0(n) \to 0 \), so
\[
\sum_n |\langle Bge_n, e_n \rangle| = \infty,
\]
and this contradicts Proposition 1.2. Thus \( A_g \) is not in the trace class.

**Remark 2.2.** The \( S^p \)-criteria for \( A_g \), \( p \geq 2 \), may also be obtained by an application of the results for Calderón-Toeplitz operators in [N].

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**References**


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