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ON BURKHOLDER'S BICONVEX-FUNCTION CHARACTERIZATION OF HILBERT SPACES

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ABSTRACT. Suppose that $\mathbf{X}$ is a real or complex Banach space with norm $|\cdot|$. Then $\mathbf{X}$ is a Hilbert space if and only if

$$E|x+Y| \geq 1$$

for all $x \in \mathbf{X}$ and all $\mathbf{X}$-valued Bochner integrable functions Y on the Lebesgue unit interval satisfying $EY = 0$ and $|Y| \geq 1$ a.e. This leads to a simple proof of the biconvex-function characterization due to Burkholder.

1. INTRODUCTION

Suppose that $\mathbf{X}$ is a real or complex Banach space with norm $|\cdot|$. Then $\mathbf{X}$ is *ζ-convex* if there is a biconvex function $\zeta: \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ such that $\zeta(0, 0) > 0$ and

$$\zeta(x, y) \leq |x+y| \quad \text{if } |x| = |y| = 1. \tag{1}$$

Biconvexity means that both $\zeta(\cdot, y)$ and $\zeta(x, \cdot)$ are convex on $\mathbf{X}$ for all y and x in $\mathbf{X}$.

The condition of ζ-convexity, discovered by Burkholder, characterizes Banach spaces with the unconditionality property for martingale differences (UMD); see [3, 6]. The condition of ζ-convexity also characterizes a class of Banach spaces important in harmonic analysis. Burkholder and McConnell [5] proved that if $\mathbf{X}$ is ζ-convex, then the Hilbert transform, defined by

$$Hf(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy,$$

is a bounded operator on the Lebesgue-Bochner space $L^p(\mathbf{R}, \mathbf{X})$ for $1 < p < \infty$, and obtained similar results for more general singular integral operators. Later Bourgain [2] proved the converse: If the Hilbert transform is a bounded operator on $L^p(\mathbf{R}, \mathbf{X})$, then $\mathbf{X}$ is ζ-convex. (See [7] for background information on the Bochner integral.)

If $\zeta: \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ is a function satisfying (1), then

$$\zeta(0, 0) \leq 1. \tag{2}$$

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To see this, take x in $\mathbf{X}$ with $|x| = 1$. Then by biconvexity and (1),

$$\zeta(0, 0) \leq \tfrac{1}{4}\{\zeta(x, x) + \zeta(x, -x) + \zeta(-x, x) + \zeta(-x, -x)\} \leq |x| = 1.$$

If $\mathbf{H}$ is a Hilbert space, there is a biconvex function ζ on $\mathbf{H} \times \mathbf{H}$ that attains the upper bound in (2). Let

$$\zeta(x, y) = 1 + (x, y) \tag{3}$$

where $(\cdot, \cdot)$ denotes the real part of the inner product of x and y. Then ζ is biconvex and $\zeta(0, 0) = 1$. Furthermore, ζ satisfies (1):

$$\begin{aligned}\zeta(x, y)^2 &\leq 1 + 2(x, y) + |x|^2|y|^2 \\ &= |x + y|^2 + (1 - |x|^2)(1 - |y|^2).\end{aligned}$$

As Burkholder observed (see [4, 6]), the converse holds.

Theorem 1. *Suppose that* $\mathbf{X}$ *is a Banach space. If there is a biconvex function* $\zeta: \mathbf{X} \times \mathbf{X} \to \mathbf{R}$ *such that* $\zeta(0, 0) = 1$ *and* (1) *is satisfied, then* $\mathbf{X}$ *is a Hilbert space.*

The proof given by Burkholder is geometrical. He reduces it in several steps to the parallelogram identity of Jordan and von Neumann. To prove Theorem 1 from a different perspective we shall consider a seemingly unrelated problem.

2. The main theorem

Let x be a point in the Banach space $\mathbf{X}$ and Y an $\mathbf{X}$-valued Bochner integrable function on the Lebesgue unit interval. Denote by EY the integral of Y on $[0, 1)$. Assume that $EY = 0$ and $|Y| \geq 1$ a.e. Then the question is: How small can $E|x + Y|$ be?

The following lemma provides a lower bound.

Lemma 2. *If* $x \in \mathbf{X}$ *and* Y *is an* $\mathbf{X}$*-valued Bochner integrable function on the Lebesgue unit interval satisfying* $EY = 0$ *and* $|Y| \geq 1$ *a.e., then*

$$E|x + Y| \geq \zeta(0, 0) \tag{4}$$

for all biconvex functions ζ *on* $\mathbf{X} \times \mathbf{X}$ *satisfying* (1).

This lemma follows from Lemma 8.1 of [6], but we shall give a direct proof here.

I. If there is a biconvex function u on $\{(x, y) \in \mathbf{X} \times \mathbf{X}: |x| \vee |y| \leq 1\}$ satisfying (1), then there is a biconvex function ζ on $\mathbf{X} \times \mathbf{X}$ such that $\zeta(0, 0) \geq u(0, 0)$ and

$$\zeta(x, y) \leq |x + y| \quad \text{if } |x| \vee |y| \geq 1. \tag{5}$$

II. If ζ is a biconvex function on $\mathbf{X} \times \mathbf{X}$ satisfying (5), then for all x, y, x', y' in $\mathbf{X}$,

$$|\zeta(x, y) - \zeta(x', y')| \leq |x - x'| + |y - y'|,$$

so ζ is continuous.

See [6] for these and related results.

Proof of Lemma 2. Take x and Y as in Lemma 2, and let ζ be a biconvex function satisfying (1). By I, we can assume that ζ satisfies (5). Replacing ζ

by the mapping $(x, y) \mapsto \zeta(x, y) \vee \zeta(-x, -y)$, if necessary, we can assume that ζ satisfies the symmetry condition

$$\zeta(x, y) = \zeta(-x, -y). \tag{6}$$

Since $|Y| \geq 1$ a.e., property (5) and Jensen's inequality applied to the continuous function $\zeta(x, \cdot)$ yield

$$E|x + Y| \geq E\zeta(x, Y) \geq \zeta(x, EY) = \zeta(x, 0).$$

From (6) and the convexity of $\zeta(\cdot, 0)$, it follows that

$$\zeta(x, 0) = \tfrac{1}{2}\{\zeta(x, 0) + \zeta(-x, 0)\} \geq \zeta(0, 0),$$

which completes the proof of Lemma 2.

In particular, if $\mathbf{X}$ is a Hilbert space, then Lemma 2 applied to the function ζ defined in (3) gives $E|x + Y| \geq 1$. A natural question is: Does $E|x+Y| \geq 1$ characterize Hilbert space?

Theorem 3. *Suppose that* $\mathbf{X}$ *is a Banach space. If*

$$E|x + Y| \geq 1$$

for all $x \in \mathbf{X}$ *and all* $\mathbf{X}$*-valued Bochner integrable functions* Y *on the Lebesgue unit interval satisfying* $EY = 0$ *and* $|Y| \geq 1$ *a.e., then* $\mathbf{X}$ *is a Hilbert space.*

3. Proofs of the theorems

In our proofs, we can assume that $\mathbf{X}$ is a Banach space over the real field. We need the following two lemmas from the theory of convex bodies. Lemma 4 is a well-known geometric characterization of Hilbert spaces; see [8, p. 144] for the proof. Lemma 5 is due to Loewner; see [1] or [8, p. 139].

Lemma 4. *Suppose that* $\mathbf{X}$ *is a two-dimensional real Banach space. Then the norm of* $\mathbf{X}$ *is generated by an inner product if and only if the unit sphere of* $\mathbf{X}$ *is an ellipse.*

Lemma 5. *If* C *is a symmetric (about the origin) closed convex curve in the plane, then there exists a unique ellipse of maximal area inscribed in* C. *The maximal inscribed ellipse touches* C *in at least four points which are symmetric pairwise.*

Proof of Theorem 3. Suppose, on the contrary, that $\mathbf{X}$ is not a Hilbert space. We shall find $x \in \mathbf{X}$ and an $\mathbf{X}$-valued simple function Y so that $EY = 0$, $|Y| \geq 1$ everywhere, but $E|x + Y| < 1$.

We can assume, without loss of generality, that the dimension of $\mathbf{X}$ is equal to two. Denote the norm of $\mathbf{X}$ by $|\cdot|$. Let S_X be the unit sphere of $\mathbf{X}$ with respect to $|\cdot|$. Then, by Lemma 5, there is an ellipse S_0 of maximal area inscribed in S_X with at least four distinct contact points which are symmetric pairwise. Denote by $\|\cdot\|$ the norm induced by S_0. After some affine transformations, we can assume that S_0 is the unit circle. Let $\pm A$ and $\pm C$ denote four contact points with no contact points in the interior of the arc $\overset{\frown}{AC}$. The existence of such points is assured by Lemma 4.

Let $\theta = \frac{1}{2}\angle AOC$, one half of the angle determined by the line segments $\overline{OA}$ and $\overline{OC}$. Here O denotes the origin of $\mathbf{X}$. By taking a rotation, if necessary, we can assume that $0 < 2\theta \leq \pi/2$, $A = (1, 0)$, and $C = (\cos 2\theta, \sin 2\theta)$. Let $D = s(\cos\theta, \sin\theta)$, where s is a positive number satisfying $|s(\cos\theta, \sin\theta)| = 1$. Accordingly, $s > 1$.

Let, for t in an interval $(-s, s)$,

$$x(t) = -t(\cos\theta, \sin\theta).$$

Let $Y\colon [0, 1) \to \mathbf{X}$ be a simple function defined by

$$Y = AI_{[0,p)} + CI_{[p,2p)} - DI_{[2p,1)} \tag{7}$$

where $p = s/2(s + \cos\theta)$ and $I_{[a,b)}$ denotes the indicator function of the interval $[a, b)$. Then it is easy to see that $EY = 0$, $|Y| = 1$ everywhere on $[0, 1)$, and

$$\begin{aligned} x(t) + Y = {} & (1 - t\cos\theta, -t\sin\theta)I_{[0,p)} \\ & + (\cos 2\theta - t\cos\theta, \sin 2\theta - t\sin\theta)I_{[p,2p)} \\ & - (s + t)(\cos\theta, \sin\theta)I_{[2p,1)}. \end{aligned}$$

Let f and g be functions defined on an interval $(-s, s)$ by

$$\begin{aligned} f(t) &= E|x(t) + Y| \\ &= p|(1 - t\cos\theta, -t\sin\theta)| + p|(\cos 2\theta - t\cos\theta, \sin 2\theta - t\sin\theta)| \\ &\quad + (1 - 2p)\frac{s + t}{s}, \\ g(t) &= p\|(1 - t\cos\theta, -t\sin\theta)\| + p\|(\cos 2\theta - t\cos\theta, \sin 2\theta - t\sin\theta)\| \\ &\quad + (1 - 2p)\frac{s + t}{s}. \end{aligned}$$

Then, for t in $(-s, s)$,

$$\begin{aligned} & f(t) \leq g(t) \quad \text{with } f(0) = g(0) = 1; \\ & g(t) = 2p(1 - 2t\cos\theta + t^2)^{1/2} + (1 - 2p)\frac{t + s}{s}; \\ & g'(t) = 2p\frac{t - \cos\theta}{(1 - 2t\cos\theta + t^2)^{1/2}} + \frac{1 - 2p}{s}. \end{aligned}$$

In particular,

$$g'(0) = \frac{(1 - s^2)\cos\theta}{s(s + \cos\theta)} < 0 \quad \text{since } s > 1 \text{ and } \frac{\pi}{4} \geq \theta > 0.$$

Since $g'(0) < 0$, we obtain $f(t) \leq g(t) < 1$ for a small positive number t. Let $x = x(t)$ for this t. Then $E|x + Y| < 1$ where Y, given by (7), satisfies $EY = 0$ and $|Y| \geq 1$ everywhere. This completes the proof of Theorem 3.

Proof of Theorem 1. Suppose that $\mathbf{X}$ is not a Hilbert space. Let ζ be a biconvex function on $\mathbf{X} \times \mathbf{X}$ satisfying (1). Then by Theorem 3, there exist a point x in $\mathbf{X}$ and a simple function Y with values in $\mathbf{X}$ such that $|Y| \geq 1$ a.e., $EY = 0$, but $E|x + Y| < 1$. Therefore, by (4), $\zeta(0, 0)$ is less than one. This completes the proof of Theorem 1.

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