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A REMARK ON CURVES COVERED BY COVERINGS

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ABSTRACT. Let $f: C' \to C$ be a covering between two smooth irreducible projective curves. Let p be a prime number. If C' is a covering of degree p of a curve of genus h and if $g(C) \geq \binom{p}{2}(h+3)+h+3-2p$, then C is a covering of degree p of a curve of genus at most h.

Let C be a smooth irreducible projective curve of genus g defined over $\mathbb{C}$.

Definition. We say that C is of type $(d; h)$ if there exists a covering $\pi: C \to E$ with $\deg(\pi) = d$ and $g(E) \leq h$.

Statement $S(d; h; g)$. *If $f: C' \to C$ is a morphism with $g(C) = g$ and if C' is of type $(d; h)$, then C is of type $(d; h)$.*

Statement $S(d; h; g)$ is proved for the cases $d = 2$ (see [1, 2]), $d = 3$ (see [1]), and $d = 4$, $g \neq 7$ (see [1]). For $h = 0$ it is almost trivial (see, e.g., [4]).

In this paper we prove

Theorem. *Let p be a prime number. Statement $S(p; h; g)$ holds for $g \geq \binom{p}{2}(h+3)+h+3-2p$.*

Remarks (from the referee). (1) There is an important difference between Statement $S(d; h; g)$ in this paper and in [1]. As a matter of fact, in [1] a curve C is said to be of type $(d; h)$ if $\deg(\pi) \leq d$ (inequality instead of equality).

(2) The referee informed me about D. Abramovich's Ph.D. Thesis, *Subvarieties of Abelian varieties and of jacobians of curves*, of which Abramovich sent me a copy. Amongst a lot of other interesting results, it contains again results concerning the topic studied in this paper. Also, from a letter, it is clear to me that he knows our theorem already.

If d is not a prime number then the situation is more complicated. Assume $d = n = an'$ for some $a, n' \in \mathbb{Z}_{\geq 2}$ and assume there exists a diagram

$$\begin{array}{ccc} C_1' & \longrightarrow & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \\ E & & \end{array}$$

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with $\deg(\pi_1) = n'$, $g(E) = h$, and $g(C_1) = g_1$ such that C_1 is not of type $(n''; h)$ for some $n'' \leq n'$. Take a covering $\tau: C \to C_1$ of degree a. Assume there exists a covering $\mu: C \to E_1$ with $\deg(\mu) \leq n$ and $g(E_1) \leq h$. Consider $(\tau; \mu): C \to C_1 \times E_1$, and let C'' be the normalization of the image of $(\tau; \mu)$. One obtains a diagram of coverings

$$\begin{tikzcd}
C \arrow[rr, "\nu"] \arrow[dr, "\tau"'] \arrow[drr, bend right, "\mu"'] & & C'' \arrow[dl, "\gamma"'] \arrow[dr, "\varepsilon"] & \\
& C_1 & & E_1
\end{tikzcd}$$

It is well known (see, e.g., [3, §1, (iii)]) that $g(C'') \leq (a-1)(n-1) + ag_1 + nh$. So if we take $g(C)$ large enough then ν is nontrivial. Also γ is nontrivial, otherwise ε gives us that C_1 is of type $(n''; h)$ for some $n'' \leq n'$. Choosing suited ramification in the covering τ (e.g., a fibre of type $(a-1)Q_1 + Q_2$ suffices), this becomes impossible. Let $\widetilde{C'}$ be the normalization of an irreducible component of $C_1' \times_{C_1} C$. Taking a covering $C' \to \widetilde{C'}$ if necessary, we obtain the following situation

$$\begin{tikzcd}
C' \arrow[r] \arrow[d, "\pi"'] & C \\
E &
\end{tikzcd}$$

with $\deg(\pi) = n$, $g(E) \leq h$, but C is not of type $(n; h)$. Moreover, we can take $g(C)$ arbitrarily large.

In [1] Abramovich and Harris prove the existence of a diagram

$$\begin{tikzcd}
C_1' \arrow[r] \arrow[d, "\pi"'] & C_1 \\
E &
\end{tikzcd}$$

with $g(C_1) = 5$, $\deg(\pi) = 3$, $g(E) = 2$, and C_1 is not of type $(n''; 2)$ for $n'' = 2$ or 3. Hence for each $a \in \mathbb{Z}_{\geq 2}$ there exists no g_0 such that $S(3a; 2; g)$ holds for $g \geq g_0$.

Proof of the theorem. Consider the situation

$$\begin{tikzcd}
C' \arrow[r, "f"] \arrow[d, "\pi"'] & C \\
E &
\end{tikzcd}$$

with $\deg(\pi) = p$, $g(E) = h$, and $g(C) = g \geq \binom{p}{2}(h+3) + h + 3 - 2p$. For $n \in \mathbb{Z}_{\geq 2}$, $D \in E^{(h+n)}$, and $P \in E$ general, the linear systems $|D - P|$ and $|D|$ on E are without fixed points. Using the fact that f_* preserves linear equivalence (see [4]) also $L := |f_*\pi^*(D)|$ and $|L - f_*\pi^*(P)| = |f_*\pi^*(D - P)|$ have no fixed points on C. In particular,

$$(*) \qquad \dim(L) \geq \dim(|L - f_*\pi^*(P)|) + 1.$$

Assume we have equality in $(*)$. Let $\phi: C \to \mathbb{P}^r$ be the morphism associated to L, let C_1 be the normalization of $\phi(C)$, and let $\rho: C \to C_1$ be the induced covering. Equality in $(*)$ implies that $\deg(\rho) = p$ and for each $P \in E$ the divisor $f_*\pi^*(P)$ is a fibre of ρ. It follows that the morphism $(\pi\,;\,\rho \circ f): C' \to E \times C_1$ factors through E. Hence E dominates C_1, so $g(C_1) \leq h$. This implies that C is of type $(p\,;\,h)$.

So now, assume we have

$$\dim(L) \geq \dim(|L - f_*\pi^*(P)|) + 2.$$

Because $\deg(\rho) < p$ now, we have that ρ is an isomorphism (here we use that p is a prime number). If $p = 2$ then for $D \in E^{(h+1)}$ general one finds that $\dim(|f_*\pi^*(D) + f_*\pi^*(P)|) \geq \dim(|f_*\pi^*(D)|) + \deg(f_*\pi^*(P))$. This implies that $|f_*\pi^*(D)|$ is nonspecial, hence $1 \leq 2h+2-g$, i.e., $g \leq 2h+1$. Since $g > 2h+1$, we obtain a contradiction. For $p \neq 2$ we are going to use Castelnuovo theory in order to obtain a similar situation. The use of Castelnuovo theory is inspired by §3 in [1].

Take $D_1 \in E^{(h+2)}$ general and $D_2 \in E^{(h+n)}$ $(n \geq 2)$ general. Take $P \in E$ general and write $f_*\pi^*(P) = P_1 + P_2 + \cdots + P_p$. Since $L_i = |f_*\pi^*(D_i)|$ is simple, we obtain that there exists $E_i \in L_i$ with $\mathrm{cd}(E_i\,;\,f_*\pi^*(P)) = P_i$. (For two effective divisors E_1 and E_2 on C, one has $E_0 = \mathrm{cd}(E_1\,;\,E_2)$ if $E_i \geq E_0$ and $\mathrm{Supp}(E_1 - E_0) \cap \mathrm{Supp}(E_2 - E_0) = \varnothing$.) Consider $|E_1 + E_2 + P_3 + \cdots + P_p| \supset (|E_1+E_2|+P_3+\cdots+P_p) \cup (|E_1+P_2+\cdots+P_p|+(E_2-P_2))$. Also $|E_1+P_2+\cdots+P_p| = |f_*\pi^*(D_1+P) - P_1|$ has no fixed points, so $|E_1 + E_2 + P_3 + \cdots + P_p|$ has no fixed points. Also $|E_1 + E_2 + P_2 + \cdots + P_p| \supset |E_2 + P_1 + \cdots + P_p| + (E_1 - P_1)$, hence $|E_1 + E_2 + P_2 + \cdots + P_p|$ has no fixed points. Finally $|E_1 + E_2 + P_1 + \cdots + P_p| = |f_*\pi^*(D_1+D_2+P)|$ has no fixed points. This implies, for $D \in E^{(2h+4+n)}$ $(n \geq 0)$ and $P \in E$ general, that one has

$$\dim|f_*\pi^*(D+P)| \geq \dim|f_*\pi^*(D)| + 3\,;$$

$$\text{there exist } E_i \in |f_*\pi^*(D+P)| \text{ for } i = 1\,,\,2$$
$$\text{with } \mathrm{cd}(E_1\,;\,f_*\pi^*(P)) = P_1,\ \mathrm{cd}(E_2\,;\,f_*\pi^*(P)) = P_1 + P_2.$$

Assume $p > 3$. Take $D_1 \in E^{(2h+5)}$ general, $D_2 \in E^{(h+2)}$ general, and $P \in E$ general, and write $f_*\pi^*(P) = P_1 + \cdots + P_p$. Take $E_1, E_2 \in |f_*\pi^*(D_1)|$ with $\mathrm{cd}(E_1\,;\,f_*\pi^*(P)) = P_1$, $\mathrm{cd}(E_2\,;\,f_*\pi^*(P)) = P_1 + P_2$, and take $E_3 \in |f_*\pi^*(D_2)|$ with $\mathrm{cd}(E_3\,;\,f_*\pi^*(P)) = P_3$. As before, for $D \in E^{(3h+7)}$ and $P \in E$ general, one has

$$\dim|f_*\pi^*(D+P)| \geq \dim|f_*\pi^*(D)| + 4\,;$$

$$\text{there exist } E_i \in |f_*\pi^*(D+P)| \text{ for } i = 1\,,\,2\,,\,3$$
$$\text{with } \mathrm{cd}(E_i\,;\,f_*\pi^*(P)) = P_1 + \cdots + P_i.$$

Continuing in this way, one finds for $D \in E^{((p-1)(h+3)-2)}$ and $P \in E$ general that

$$\dim|f_*\pi^*(D+P)| \geq \dim|f_*\pi^*(D)| + p.$$

This is only possible if $|f_*\pi^*(D)|$ is nonspecial on C. But $\deg(f_*\pi^*(D)) = p(p-1)(h+3) - 2p$ and $\dim(|f_*\pi^*(D)|) \geq 1 + 2(h+3) + 3(h+3) + \cdots + (p-1)(h+3) = \binom{p}{2}(h+3) - h - 2$. So $\binom{p}{2}(h+3) - h - 2 \leq p(p-1)(h+3) - 2p - g$, hence $g \leq \binom{p}{2}(h+3) - 2p + h + 2$. This gives us a contradiction.

Remark. Using the arguments of [1], the situation is related to the existence of an abelian variety A of dimension h contained in $C^{(ph)}$. Then Theorem 2 in [1] gives us a bound of order p^2h^2, which is worse than ours.

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