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CRINKLED FUNCTIONS AND INTERSECTIONS WITH POLYNOMIALS

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ABSTRACT. We prove that if $\|f^{(n+1)}\|$ is large and $\|f^{(n)}\|_\infty$ is bounded, then there is a polynomial p of degree n such that $f(t) = p(t)$ has many solutions.

A classical theorem of Banach (Natanson [1]) says that if $n_f(\alpha) =$ the number of sign changes of $\{f(t)-\alpha\}$ on $[0,1]$ and f is a continuous function of bounded variation, then n_f is measurable and

$$\int_{-\infty}^{\infty} n_f(\alpha)\, d\alpha = TV(f). \tag{1}$$

Assume now that $|f(t)| \le M$ on $[0,1]$ and f is differentiable. Let $N(f) =$ ess $\sup_{\alpha \in R} n_f(\alpha)$. Since $n_f(\alpha) = 0$ for $|\alpha| > M$ we obtain from (1)

$$\int_0^1 |f'(t)|\, dt = \int_{-M}^{M} n_f(\alpha)\, d\alpha \le 2MN(f). \tag{2}$$

We may state this as an intersection theorem.

Theorem A. *Let $f \in C^1[0, 1]$ and k be a positive integer. If $|f(t)| \le M$ and $\|f'\|_1 > 2(k-1)M$ then*

$$N(f) \ge k \tag{3}$$

and there is an $\alpha \in R$ so that the equation $f(t) = \alpha$ has at least k distinct solutions on $[0, 1]$. The constant $2(k-1)M$ cannot be replaced by any smaller number and have (3) hold.

Proof. Combining the hypothesis with (2) yields $N(f) > k-1$. Since $N(f)$ is an integer, (3) holds. A piecewise linear function with $f(0) = -M$ and alternate maximum values M and minimum values $-M$, shows that the constant cannot be improved.

This simple result quantifies the intuitive notion that a bounded crinkled function graph must agree with some horizontal line often.

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Schrader in his researches on the compactness problem for nth order differential equations asked if there was some version of Theorem A whose hypothesis involved $f^{(n+1)}$ and whose conclusion was about intersection with polynomials of degree n. It is our purpose to supply such a theorem.

In some sense, we are looking for a converse of Rolle's theorem. For example, if f agrees with $p(x) = \alpha x + \beta$, a polynomial of degree 1 often, then by Rolle's theorem $f' - \alpha$ has many zeroes. By applying Theorem A to f' we would have this conclusion if $|f'| \leq M$ and $\|f''\|$ is large. But conversely having $\|f''\|$ large and $|f'| \leq M$ only gets $f' - \alpha$ with lots of zeros. Integration does not give $f(x) = \alpha x + \beta$ often. Nevertheless we are able to prove an exact replica of Theorem A in the general case.

Theorem B. *For every nonnegative integer n there is a constant $C_n > 0$ so that for $f \in AC^{(n)}(0, 1)$, the inequality*

$$\|f^{(n+1)}\|_1 > C_n \cdot (k-1) \cdot \|f^{(n)}\|_\infty \tag{4}$$

implies the existence of a polynomial p of degree $\leq n$ so that the equation $f(t) = p(t)$ has at least k distinct solutions on $[0, 1]$.

Let $f \in AC^{(n)}(0, 1)$, and put

$$F_n(t, u) = \frac{1}{n!} \int_u^t (t-s)^n f^{(n+1)}(s)\, ds = f(t) - \sum_{i=0}^{n} \frac{f^{(i)}(u)}{i!}(t-u)^i \tag{5}$$

for every $u, t \in [0, 1]$. Then for any fixed u, the equation $f_n(t, u) = \alpha$ is the equation $f(t) = p(t)$ where p is a polynomial of degree n in the variable t. If $n_{(f,n)}(\alpha)$ denotes the number of sign changes in $F_n(t, u) = \alpha$, then Theorem A reads

$$\int_{-\infty}^{\infty} n_{(f,n)}(\alpha)\, d\alpha = \int_0^1 \left| \frac{d}{dt} F_n(t, u) \right| dt = \frac{1}{(n-1)!} \int_0^1 \left| \int_u^t (t-s)^{n-1} f^{(n+1)}(s)\, ds \right| dt.$$

We choose to integrate with respect to u also to get

$$\int_0^1 \int_{-\infty}^{\infty} n_{(f,n)}(\alpha)\, d\alpha\, du = \frac{1}{(n-1)!} \int_0^1 \int_1^1 \left| \int_u^t (t-s)^{n-1} f^{(n+1)}(s)\, ds \right| dt\, du. \tag{6}$$

We want to relate this integral to $\|f^{(n+1)}\|_1$. To do this define

$$Tg = \int_0^1 \int_0^1 \left| \int_u^t (t-s)^{n-1} g(s)\, ds \right| dt\, du. \tag{7}$$

Proposition. *Let $B = \{g \in L^2[0, 1] \mid \|g\|_2 = 1\}$ and $d_n = \inf_{g \in B} Tg$. Then there is a $g_0 \in B$ such that $d_n = Tg_0$ and $d_n > 0$.*

Proof. Let $g_i \in B$ such that $Tg_i \to d_n$. By weak compactness of B we may assume that g_i converges weakly to $g_0 \in B$, i.e., $\lim_{i \to \infty} \int_0^1 g_i f\, ds = \int_0^1 g_0 f\, ds$ for all $f \in L^2[0, 1]$. Now

$$\int_u^t (t-s)^{n-1} g_i(s)\, ds = \int_0^1 (t-s)^{n-1} \chi_{[u,t]} g_i(s)\, ds$$

and so

$$\int_u^t (t-s)^{n-1} g_i(s)\,ds \to \int_u^t (t-s)^{n-1} g_0(s)\,ds$$

for every $t, u \in [0, 1]$. Since $|\int_u^t (t-s)^{n-1} g_i(s)\,ds| \le \|g_i\|_1 \le \|g_i\|_2 = 1$, by the Lebesgue dominated convergence theorem we have $\lim_i Tg_i = Tg_0 = d_n$. We now show that $d_n > 0$.

If $d_n = 0$ then by (7)

$$\int_u^t (t-s)^{n-1} g_0(s)\,ds = 0 \qquad (t, u \in [0, 1]).$$

Since $g_0 \in L^2 \subset L^1$, there is $f \in AC^{(n)}(0, 1)$ such that $f^{(n+1)} = g_0$ a.e. Then

$$f(t) - \sum_{i=0}^{n-1} \frac{f^{(i)}(u)}{i!}(t-u)^i = \frac{1}{(n-1)!}\int_u^t (t-s)^{n-1} f^{(n+1)}(s)\,ds = 0$$

for every t and u and thus f is a polynomial of degree $\le n$. Therefore $g_0 = f^{(n+1)} = 0$ a.e., contradicting the equation $\|g_0\|_2 = 1$.

Proof of Theorem B. For $n = 0$ Theorem B is Theorem A so we may assume $n \ge 1$. Let $C_n = 4/d_n$ and $\|f^{(n+1)}\|_1 > C_n \cdot (k-1) \cdot \|f^{(n)}\|_\infty$. Then by the Cauchy-Schwarz inequality

$$\|f^{(n+1)}\|_1 \le \|f^{(n+1)}\|_2 \quad \text{so} \quad \|f^{(n+1)}\|_2 > C_n \cdot (k-1) \cdot \|f^{(n)}\|_\infty. \tag{8}$$

Combining (8), (6), and the proposition, we obtain

$$\int_0^\infty \int_{-\infty}^\infty n_{(f,n)}(\alpha)\,d\alpha\,du \ge \frac{d_n}{n!}\|f^{(n+1)}\|_2 > \frac{4}{n!}\|f^{(n)}\|_\infty \cdot (k-1). \tag{9}$$

Since, by (5),

$$F_n(t, u) = \frac{f^{(n)}(\xi)}{n!}(t-u)^n - \frac{f^{(n)}(u)}{n!}(t-u)^n,$$

we have $|F_n(t, u)| \le \frac{2}{n!}\|f^{(n)}\|_\infty$. The left-hand side of (9) may be estimated by $\frac{4}{n!}\|f^{(n)}\|_\infty \cdot \operatorname{ess\,sup} n_{(f,n)}(\alpha)$. Thus $\operatorname{ess\,sup} n_{(f,n)} > k-1$, and the result follows.

Results complementary to ours are given in Agronsky et al. [2] who show that if intersections with polynomials are few, then certain derivatives must be monotone on subintervals.

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References

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