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# ON MAXIMAL $k$-IDEALS OF SEMIRINGS

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ABSTRACT. For a semiring $S$ with commutative addition, conditions are considered such that $S$ has nontrivial $k$-ideals or maximal $k$-ideals, among others, by the help of the congruence class semiring $S/A$ defined by an ideal $A$ of $S$. Moreover, all maximal $k$-ideals of the semiring of nonnegative integers are described.

## 1. PRELIMINARIES

A *semiring* $S$ is defined as an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for all $a, b, c \in S$. A semiring $S$ may have an *identity* 1 [a zero $o$], defined by $1a = a1 = a[o + a = a + o = a]$ for all $a \in S$. If there is an element $O \in S$ satisfying $Oa = aO = O$ for all $a \in S$, it is called *multiplicatively absorbing* or simply *absorbing*. Such an element satisfies $O + O = O$, but it need not be a zero of $S$, whereas a zero $o$ of $S$ need not even satisfy $oo = o$. Clearly, a semiring has an *absorbing zero* iff it has elements $O$ and $o$ which coincide.

A subset $A \neq \varnothing$ of a semiring $S$ is called an *ideal* of $S$ iff $a + b \in A$, $sa \in A$, and $as \in A$ hold for all $a, b \in A$ and all $s \in S$. An ideal $A$ of $S$ is called *proper* iff $A \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal $A$ is called *maximal* iff there is no ideal $B$ of $S$ satisfying $A \subset B \subset S$. Obviously, a semiring $S$ contains an ideal $A$ consisting of one element iff $S$ has an absorbing element $O$, and then $A = \{O\}$ is the only ideal of this kind. Finally, an ideal $A$ of $S$ is called *trivial*, iff $A = S$ holds or $A = \{O\}$, the latter clearly if $S$ has an absorbing element. To deal with both cases simultaneously, we introduce the notion $S'$ by $S' = S \backslash \{O\}$ if $S$ has an absorbing element, and $S' = S$ otherwise.

*In this paper we only consider semirings $S$ for which $(S, +)$ is commutative.* If also $(S, \cdot)$ is commutative, $S$ is called a *commutative semiring*. Moreover, to avoid trivial exceptions, *each semiring $S$ is assumed to have at least two elements.*

Using only commutativity of addition, the following concepts and statements, essentially due to [1, 2, 4], are well known. For each ideal $A$ of a semiring $S$

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the *k-closure* $\overline{A}$ of $A$ defined by

$$\overline{A} = \{\overline{a} \in S | \overline{a} + a_1 = a_2 \text{ for some } a_i \in A\}$$

is an ideal of $S$ satisfying $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$. An ideal $A$ of $S$ is called a *k-ideal* of $S$ iff $A = \overline{A}$ holds. Clearly, $S$ is a $k$-ideal for each semiring $S$; however, if $S$ has an absorbing element $O$, the ideal $\{O\}$ need not be a $k$-ideal of $S$. There are examples for $\{O\} \subset \overline{\{O\}} \subset S$ and $\{O\} \subset \overline{\{O\}} = S$, whereas $\{O\} = \overline{\{O\}}$ holds if $O$ is an absorbing zero of $S$. A $k$-ideal $A \subset S$ is called a *maximal k-ideal* of $S$ if there is no $k$-ideal $B$ of $S$ satisfying $A \subset B \subset S$. Note that a maximal $k$-ideal of $S$ need not be a maximal ideal of $S$ (cf. Remark 4.2).

Moreover, each ideal $A$ of $S$ defines a congruence $\rho_A$ on $(S, +, \cdot)$ by

$$\rho_A = \{(x, y) \in S \times S | x + a_1 = y + a_2 \text{ for some } a_i \in A\}.$$

The corresponding congruence class semiring $S/\rho_A$, consisting of the classes $x\rho_A$, is also denoted by $S/A$. The $k$-closure $\overline{A}$ of $A$ is such a congruence class, and $\overline{A}$ is the absorbing zero of $S/A$, regardless of whether $S$ has a zero $o$ or an absorbing element $O$ (which implies $o\rho_A = \overline{A}$ or $O\rho_A = \overline{A}$, respectively). Moreover, $\rho_A$ and $\rho_{\overline{A}}$, and hence $S/A$ and $S/\overline{A}$ coincide.

## 2. Maximal $k$-ideals

**Theorem 2.1.** *Let $S$ be a semiring such that $S = (a_1, \dots, a_n)$ is a finitely generated ideal of $S$. Then each proper $k$-ideal $A$ of $S$ is contained in a maximal $k$-ideal of $S$.*

*Proof.* Let $\mathfrak{B}$ be the set of all $k$-ideals $B$ of $S$ satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i | i \in I\}$ in $\mathfrak{B}$. One easily checks that $B = \bigcup_{i \in I} B_i$ is a $k$-ideal of $S$, and $S = (a_1, \dots, a_n)$ implies $B \neq S$, and hence $B \in \mathfrak{B}$. So by Zorn's lemma, $\mathfrak{B}$ has a maximal element as we were to show.

**Corollary 2.2.** *Let $S$ be a semiring with identity $1$. Then each proper $k$-ideal of $S$ is contained in a maximal $k$-ideal of $S$.*

The proof is immediate by $S = (1)$.

**Definition 2.3.** A semiring $S$ is said to satisfy condition (C) iff for all $a \in S'$ and all $s \in S$ there are $s_1, s_2 \in S$ such that

$$s + s_1 a = s_2 a$$

holds. Clearly, if $S$ has an identity 1, then (C) is equivalent to the following condition (C′):

$$1 + s_1 a = s_2 a$$

holds for each $a \in S'$ and suitable $s_1, s_2 \in S$.

**Example 2.4.** Let $P$ be the set of all nonnegative rational numbers. Then $(P, +, \cdot)$ with the usual operations, as well as $(P', +, \cdot)$, are semirings with 1 as identity satisfying condition (C′). The same is true, more generally, for each positive cone $P$ of a totally ordered skew-field (cf. [3, Chapter VI]).

**Example 2.5.** Let $\mathbb{N}$ be the set of all nonnegative integers. Define $a + b = \max\{a, b\}$, and denote by $a \cdot b$ the usual multiplication. Then $(\mathbb{N}, +, \cdot)$ is a

semiring with 1 as identity, which satisfies (C′) since $1 + a = a$ holds for all $a \in S'$.

**Lemma 2.6.** *If a semiring $S$ with an absorbing zero $O$ satisfies condition* (C), *then $ab = O$ for $a, b \in S$ implies $a = O$ or $b = O$.*

*Proof.* By way of contradiction, assume $ab = O$ and $a \neq O \neq b$. Then $s + s_1 a = s_2 a$, according to (C), yields $sb + s_1 ab = s_2 ab$, i.e., $sb = O$ for all $s \in S$. Consequently, $x + s_3 b = s_4 b$ implies $x = O$ for all $s_3, s_4 \in S$, which contradicts (C) applied to the element $b \in S'$.

**Theorem 2.7.** *Let $S$ be a semiring. Then condition* (C) *implies that $S$ contains only trivial $k$-ideals. The converse is true if $(S, \cdot)$ is commutative, and, provided that $S$ has an absorbing element $O$, $Sa = \{sa | s \in S\} \neq \{O\}$ holds for all $a \in S'$.*

*Proof.* Assume that $S$ satisfies (C). Let $A$ be a $k$-ideal of $S$ which contains at least one element $a \in S'$. Then $s + s_1 a = s_2 a$, according to (C), implies $s \in A$ for each $s \in S$, i.e., $A = S$. For the converse, our supplementary assumptions on $S$ yield that $Sa$ is an ideal of $S$ and that $Sa \neq \{O\}$ holds for each $a \in S'$ if $S$ has an absorbing element $O$. Now assume that $S$ has only trivial $k$-ideals. Then the $k$-ideal $\overline{Sa}$ coincides with $S$ for each $a \in S'$, regardless of whether $S$ has an element $O$ or not. Now,

$$\overline{Sa} = \{s \in S | s + s_1 a = s_2 a \text{ for some } s_i \in S\} = S$$

states that $S$ satisfies condition (C).

**Corollary 2.8.** *Let $S$ be a commutative semiring with identity. Then $S$ has only trivial $k$-ideals iff it satisfies condition* (C′).

*Proof.* It was already stated that (C′) is equivalent to (C) if $S$ has an identity 1, and $a = 1a \in Sa$ implies $Sa \neq \{O\}$ for all $a \in S'$ in the case that $S$ has an absorbing element $O$. Hence the corollary follows from Theorem 2.7.

**Theorem 2.9.** *Let $S$ be a commutative semiring with identity* 1 *and $A$ a proper $k$-ideal of $S$. Then $A$ is maximal iff the semiring $S/A = S/\rho_A$ satisfies condition* (C′).

*Proof.* Suppose $A$ is a maximal $k$-ideal of $S$. Then $A$ is the absorbing zero of $S/A$ and $1\rho_A$ is its identity. Consider any $c\rho_A \in (S/A)'$. Then $c \notin A$ holds, and the smallest ideal $B$ of $S$ containing $c$ and $A$ consists of all elements $sc$, $a$, and $sc + a$ for $s \in S$ and $a \in A$. From $A \subset B$ it follows $\overline{B} = S$, and hence $1 + b_1 = b_2$ for suitable elements $b_1, b_2 \in B$. To avoid the discussion of different cases, we add $1c + a$ with an arbitrary element $a \in A$ to $1 + b_1 = b_2$ and obtain

$$1 + s_1 c + a_1 = s_2 c + a_2, \quad \text{i.e., } 1\rho_A + (s_1\rho_A)(c\rho_A) = (s_2\rho_A)(c\rho_A)$$

for suitable $s_i \in S$ and $a_i \in A$. This shows that $S/A$ satisfies (C′).

Conversely, assume (C′) for $S/A$, and let $B$ be a $k$-ideal of $S$ satisfying $A \subset B$. Then there is an element $c \in B \backslash A$, and $c\rho_A \in (S/A)'$ yields $(1 + s_1 c)\rho_A = (s_2 c)\rho_A$ for suitable elements $s_i \in S$ by (C′). Hence $1 + s_1 c + a_1 = s_2 c + a_2$ holds for some $a_i \in A$, i.e., $1 + b_1 = b_2$ for $b_1, b_2 \in B$. This shows $\overline{B} = S$ and that $A$ is a maximal $k$-ideal of $S$.

## 3. Completely prime $k$-ideals

Recall that an ideal $A$ of a semiring $S$ is called *completely prime* (cf., e.g., [5]) iff $ab \in A$ implies $a \in A$ or $b \in A$ for all $a, b \in S$.

**Proposition 3.1.** *Let $S$ be a commutative semiring with identity. Then each maximal $k$-ideal $A$ of $S$ is completely prime.*

*Proof.* By Theorem 2.9, the semiring $S/A$ satisfies the condition (C′) and hence (C). Since $S/A$ has $A$ as its absorbing zero, we can apply Lemma 2.6 and obtain that $S/A$ has no zero-divisors. Hence $a\rho_A \neq A$ and $b\rho_A \neq A$ imply $(ab)\rho_A \neq A$, i.e., $a \notin A$ and $b \notin A$ imply $ab \notin A$ as we were to show.

*Concerning the converse of Proposition* 3.1, *we show that a completely prime ideal $A$ of a commutative semiring $S$ with identity need not be a $k$-ideal, and if it is one, $A$ need not be a maximal $k$-ideal of $S$.*

**Example 3.2.** Let $S$ be the set of all real numbers $a$ satisfying $0 < a \leq 1$, and define $a + b = a \cdot b = \min\{a, b\}$ for all $a, b \in S$. Then $(S, +, \cdot)$ is easily checked to be a commutative semiring with 1 as identity. Each real number $r$ such that $0 < r < 1$ defines an ideal $A = \{a \in S | a \leq r\}$ of $S$ which is obviously completely prime. However, $r + 1 = r$ together with $r \in A$ and $1 \notin A$ show that $A$ is not a $k$-ideal of $S$. The same is true if one includes 0 in these considerations (in this case 0 is an absorbing element but not a zero of $S \cup \{0\}$), but also if one adjoins 0 as an absorbing zero to $S$ (cf., e.g., [7, Lemma 1.3]).

**Example 3.3.** The polynomial ring $\mathbb{Z}[x]$ over the ring $\mathbb{Z}$ of integers contains the subsemiring

$$S = \mathbb{N}[x] = \left\{ f(x) = \sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{N} \right\},$$

which is clearly commutative and has $1 \in \mathbb{N}$ as its identity. The ideal $A = (x)$ of $S$ consists of all $f(x) \in S$ such that $a_0 = 0$ holds. Obviously, $A$ is completely prime and a $k$-ideal of $S$. Now consider the set $B$ consisting of all $f(x) \in S$ for which $a_0$ is divisible by 2. Clearly, $B$ is a $k$-ideal of $S$, and $A \subset B \subset S$ shows that $A$ is not a maximal $k$-ideal.

## 4. Maximal $k$-ideals of $\mathbb{N}$

In this section we consider the semiring $(\mathbb{N}, +, \cdot)$ of nonnegative integers with respect to their usual operations.

**Proposition 4.1.** *The semiring $\mathbb{N}$ has exactly the $k$-ideals $(a) = \{na | n \in \mathbb{N}\}$ for each $a \in \mathbb{N}$. Consequently, the maximal $k$-ideals of $\mathbb{N}$ are given by $(p)$ for each prime number $p$.*

*Proof.* Obviously, each ideal $(a)$ of $\mathbb{N}$ is a $k$-ideal. Now assume that $A \neq (0)$ is a $k$-ideal of $\mathbb{N}$. Let $a$ be the smallest positive integer contained in $A$, and $b$ any element of $A$. Then $b = qa + r$ holds for some $q \in \mathbb{N}'$ and $r \in \mathbb{N}$ satisfying $0 \leq r < a$. Since $r$ belongs to the $k$-ideal $A$, it follows that $r = 0$, and, hence, $A = (a)$. The last statement follows since $(a) \subseteq (b)$ holds iff $b$ divides $a$.

*Remark* 4.2. None of the maximal $k$-ideals $(p)$ of $\mathbb{N}$ is a maximal ideal of $\mathbb{N}$. This follows since each ideal $A = (p)$ is properly contained in the proper ideal $B = \{b \in \mathbb{N} | b \geq p\}$ of $\mathbb{N}$.

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