ON MAXIMAL $k$-IDEALS OF SEMIRINGS

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Abstract. For a semiring $S$ with commutative addition, conditions are considered such that $S$ has nontrivial $k$-ideals or maximal $k$-ideals, among others, by the help of the congruence class semiring $S/A$ defined by an ideal $A$ of $S$. Moreover, all maximal $k$-ideals of the semiring of nonnegative integers are described.

1. Preliminaries

A semiring $S$ is defined as an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. A semiring $S$ may have an identity $1$ [a zero 0], defined by $1a = a1 = a[0 + a = a + 0 = a]$ for all $a \in S$. If there is an element $O \in S$ satisfying $Oa = aO = O$ for all $a \in S$, it is called multiplicatively absorbing or simply absorbing. Such an element satisfies $O + O = O$, but it need not be a zero of $S$, whereas a zero $o$ of $S$ need not even satisfy $oo = o$. Clearly, a semiring has an absorbing zero iff it has elements $O$ and $o$ which coincide.

A subset $A \neq \emptyset$ of a semiring $S$ is called an ideal of $S$ iff $a + b \in A$, $sa \in A$, and $as \in A$ hold for all $a, b \in A$ and all $s \in S$. An ideal $A$ of $S$ is called proper iff $A \subset S$ holds, where $\subset$ denotes proper inclusion, and a proper ideal $A$ is called maximal iff there is no ideal $B$ of $S$ satisfying $A \subset B \subset S$. Obviously, a semiring $S$ contains an ideal $A$ consisting of one element iff $S$ has an absorbing element $O$, and then $A = \{O\}$ is the only ideal of this kind. Finally, an ideal $A$ of $S$ is called trivial, iff $A = S$ holds or $A = \{O\}$, the latter clearly if $S$ has an absorbing element. To deal with both cases simultaneously, we introduce the notion $S'$ by $S' = S\{O\}$ if $S$ has an absorbing element, and $S' = S$ otherwise.

In this paper we only consider semirings $S$ for which $(S, +)$ is commutative. If also $(S, \cdot)$ is commutative, $S$ is called a commutative semiring. Moreover, to avoid trivial exceptions, each semiring $S$ is assumed to have at least two elements.

Using only commutativity of addition, the following concepts and statements, essentially due to [1, 2, 4], are well known. For each ideal $A$ of a semiring $S$.
the *k*-closure $\overline{A}$ of $A$ defined by

$$\overline{A} = \{\overline{a} \in S | \overline{a} + a_1 = a_2 \text{ for some } a_i \in A\}$$

is an ideal of $S$ satisfying $A \subseteq \overline{A}$ and $\overline{A} = \overline{A}$. An ideal $A$ of $S$ is called a *k-ideal* of $S$ iff $A = \overline{A}$ holds. Clearly, $S$ is a $k$-ideal for each semiring $S$; however, if $S$ has an absorbing element $O$, the ideal $\{O\}$ need not be a $k$-ideal of $S$. There are examples for $\{O\} \subset \{\overline{O}\} \subset S$ and $\{O\} \subset \{\overline{O}\} = S$, whereas $\{O\} = \{\overline{O}\}$ holds if $O$ is an absorbing zero of $S$. A $k$-ideal $A \subset S$ is called a *maximal k-ideal* of $S$ if there is no $k$-ideal $B$ of $S$ satisfying $A \subset B \subset S$. Note that a maximal $k$-ideal of $S$ need not be a maximal ideal of $S$ (cf. Remark 4.2).

Moreover, each ideal $A$ of $S$ defines a congruence $\rho_A$ on $(S, +, \cdot)$ by

$$\rho_A = \{(x, y) \in S \times S | x + a_1 = y + a_2 \text{ for some } a_i \in A\}.$$ 

The corresponding congruence class semiring $S/\rho_A$, consisting of the classes $x \rho_A$, is also denoted by $S/A$. The $k$-closure $\overline{A}$ of $A$ is such a congruence class, and $\overline{A}$ is the absorbing zero of $S/A$, regardless of whether $S$ has a zero $o$ or an absorbing element $O$ (which implies $o \rho_A = \overline{A}$ or $O \rho_A = \overline{A}$, respectively). Moreover, $\rho_A$ and $\rho_{\overline{A}}$, and hence $S/A$ and $S/\overline{A}$ coincide.

### 2. Maximal $k$-ideals

**Theorem 2.1.** Let $S$ be a semiring such that $S = (a_1, \ldots, a_n)$ is a finitely generated ideal of $S$. Then each proper $k$-ideal $A$ of $S$ is contained in a maximal $k$-ideal of $S$.

**Proof.** Let $\mathcal{B}$ be the set of all $k$-ideals $B$ of $S$ satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i | i \in I\}$ in $\mathcal{B}$. One easily checks that $B = \bigcup_{i \in I} B_i$ is a $k$-ideal of $S$, and $S = (a_1, \ldots, a_n)$ implies $B \neq S$, and hence $B \in \mathcal{B}$. So by Zorn's lemma, $\mathcal{B}$ has a maximal element as we were to show.

**Corollary 2.2.** Let $S$ be a semiring with identity $1$. Then each proper $k$-ideal of $S$ is contained in a maximal $k$-ideal of $S$.

The proof is immediate by $S = \{1\}$.

**Definition 2.3.** A semiring $S$ is said to satisfy condition (C) iff for all $a \in S'$ and all $s \in S$ there are $s_1, s_2 \in S$ such that

$$s + s_1a = s_2a$$

holds. Clearly, if $S$ has an identity $1$, then (C) is equivalent to the following condition (C'):

$$1 + s_1a = s_2a$$

holds for each $a \in S'$ and suitable $s_1, s_2 \in S$.

**Example 2.4.** Let $P$ be the set of all nonnegative rational numbers. Then $(P, +, \cdot)$ with the usual operations, as well as $(P', +, \cdot)$, are semirings with $1$ as identity satisfying condition (C'). The same is true, more generally, for each positive cone $P$ of a totally ordered skew-field (cf. [3, Chapter VI]).

**Example 2.5.** Let $\mathbb{N}$ be the set of all nonnegative integers. Define $a + b = \max\{a, b\}$, and denote by $a \cdot b$ the usual multiplication. Then $(\mathbb{N}, +, \cdot)$ is a
semiring with 1 as identity, which satisfies (C') since \( 1 + a = a \) holds for all \( a \in S' \).

**Lemma 2.6.** If a semiring \( S \) with an absorbing zero \( O \) satisfies condition (C), then \( ab = O \) for \( a, b \in S \) implies \( a = O \) or \( b = O \).

**Proof.** By way of contradiction, assume \( ab = O \) and \( a \neq O \neq b \). Then \( s + s_1a = s_2a \), according to (C), yields \( sb + s_1ab = s_2ab \), i.e., \( sb = O \) for all \( s \in S \). Consequently, \( x + s_3b = s_4b \) implies \( x = O \) for all \( s_3, s_4 \in S \), which contradicts (C) applied to the element \( b \in S' \).

**Theorem 2.7.** Let \( S \) be a semiring. Then condition (C) implies that \( S \) contains only trivial \( k \)-ideals. The converse is true if \( (S, \cdot) \) is commutative, and, provided that \( S \) has an absorbing element \( O \), \( Sa = \{ sa | s \in S \} \neq \{ O \} \) holds for all \( a \in S' \).

**Proof.** Assume that \( S \) satisfies (C). Let \( A \) be a \( k \)-ideal of \( S \) which contains at least one element \( a \in S' \). Then \( s + s_1a = s_2a \), according to (C), implies \( s \in A \) for each \( s \in S \), i.e., \( A = S \). For the converse, our supplementary assumptions on \( S \) yield that \( Sa \) is an ideal of \( S \) and that \( Sa \neq \{ O \} \) holds for each \( a \in S' \) if \( S \) has an absorbing element \( O \). Now assume that \( S \) has only trivial \( k \)-ideals. Then the \( k \)-ideal \( Sa \) coincides with \( S \) for each \( a \in S' \), regardless of whether \( S \) has an element \( O \) or not. Now,

\[
Sa = \{ s \in S | s + s_1a = s_2a \text{ for some } s_i \in S \} = S
\]

states that \( S \) satisfies condition (C).

**Corollary 2.8.** Let \( S \) be a commutative semiring with identity. Then \( S \) has only trivial \( k \)-ideals if it satisfies condition (C').

**Proof.** It was already stated that (C') is equivalent to (C) if \( S \) has an identity 1, and \( a = 1a \in Sa \) implies \( Sa \neq \{ O \} \) for all \( a \in S' \) in the case that \( S \) has an absorbing element \( O \). Hence the corollary follows from Theorem 2.7.

**Theorem 2.9.** Let \( S \) be a commutative semiring with identity 1 and \( A \) a proper \( k \)-ideal of \( S \). Then \( A \) is maximal iff the semiring \( S/A = S/\rho_A \) satisfies condition (C').

**Proof.** Suppose \( A \) is a maximal \( k \)-ideal of \( S \). Then \( A \) is the absorbing zero of \( S/A \) and \( 1\rho_A \) is its identity. Consider any \( c\rho_A \in (S/A)' \). Then \( c \notin A \) holds, and the smallest ideal \( B \) of \( S \) containing \( c \) and \( A \) consists of all elements \( sc, a \), and \( sc + a \) for \( s \in S \) and \( a \in A \). From \( A \subseteq B \) it follows \( B = S \), and hence \( 1 + b_1 = b_2 \) for suitable elements \( b_1, b_2 \in B \). To avoid the discussion of different cases, we add \( 1c + a \) with an arbitrary element \( a \in A \) to \( 1 + b_1 = b_2 \) and obtain

\[
1 + s_1c + a_1 = s_2c + a_2, \quad \text{i.e., } \quad 1\rho_A + (s_1\rho_A)(c\rho_A) = (s_2\rho_A)(c\rho_A)
\]

for suitable \( s_i \in S \) and \( a_i \in A \). This shows that \( S/A \) satisfies (C').

Conversely, assume (C') for \( S/A \), and let \( B \) be a \( k \)-ideal of \( S \) satisfying \( A \subseteq B \). Then there is an element \( c \in B \setminus A \), and \( c\rho_A \in (S/A)' \) yields \( (1 + s_1c)\rho_A = (s_2c)\rho_A \) for suitable elements \( s_1 \in S \) by (C'). Hence \( 1 + s_1c + a_1 = s_2c + a_2 \) holds for some \( a_i \in A \), i.e., \( 1 + b_1 = b_2 \) for \( b_1, b_2 \in B \). This shows \( B = S \) and that \( A \) is a maximal \( k \)-ideal of \( S \).
3. Completely prime k-ideals

Recall that an ideal \( A \) of a semiring \( S \) is called completely prime (cf., e.g., [5]) iff \( ab \in A \) implies \( a \in A \) or \( b \in A \) for all \( a, b \in S \).

**Proposition 3.1.** Let \( S \) be a commutative semiring with identity. Then each maximal k-ideal \( A \) of \( S \) is completely prime.

**Proof.** By Theorem 2.9, the semiring \( S/A \) satisfies the condition \((C')\) and hence \((C)\). Since \( S/A \) has \( A \) as its absorbing zero, we can apply Lemma 2.6 and obtain that \( S/A \) has no zero-divisors. Hence \( ap_A \neq A \) and \( bp_A \neq A \) imply \((ab)p_A \neq A\), i.e., \( a \notin A \) and \( b \notin A \) imply \( ab \notin A \) as we were to show.

Concerning the converse of Proposition 3.1, we show that a completely prime ideal \( A \) of a commutative semiring \( S \) with identity need not be a k-ideal, and if it is one, \( A \) need not be a maximal k-ideal of \( S \).

**Example 3.2.** Let \( S \) be the set of all real numbers \( a \) satisfying \( 0 < a \leq 1 \), and define \( a + b = a \cdot b = \min\{a, b\} \) for all \( a, b \in S \). Then \((S, +, \cdot)\) is easily checked to be a commutative semiring with 1 as identity. Each real number \( r \) such that \( 0 < r < 1 \) defines an ideal \( A = \{a \in S|a < r\} \) of \( S \) which is obviously completely prime. However, \( r + 1 = r \) together with \( r \in A \) and \( 1 \notin A \) show that \( A \) is not a k-ideal of \( S \). The same is true if one includes 0 in these considerations (in this case 0 is an absorbing element but not a zero of \( S \cup \{0\} \)), but also if one adjoins 0 as an absorbing zero to \( S \) (cf., e.g., [7, Lemma 1.3]).

**Example 3.3.** The polynomial ring \( \mathbb{Z}[x] \) over the ring \( \mathbb{Z} \) of integers contains the subsemiring

\[
S = \mathbb{N}[x] = \left\{ f(x) = \sum_{i=0}^{n} a_i x^i | a_i \in \mathbb{N} \right\},
\]

which is clearly commutative and has 1 as its identity. The ideal \( A = (x) \) of \( S \) consists of all \( f(x) \in S \) such that \( a_0 = 0 \) holds. Obviously, \( A \) is completely prime and a k-ideal of \( S \). Now consider the set \( B \) consisting of all \( f(x) \in S \) for which \( a_0 \) is divisible by 2. Clearly, \( B \) is a k-ideal of \( S \), and \( A \subset B \subset S \) shows that \( A \) is not a maximal k-ideal.

4. Maximal k-ideals of \( \mathbb{N} \)

In this section we consider the semiring \((\mathbb{N}, +, \cdot)\) of nonnegative integers with respect to their usual operations.

**Proposition 4.1.** The semiring \( \mathbb{N} \) has exactly the k-ideals \( (a) = \{na|n \in \mathbb{N}\} \) for each \( a \in \mathbb{N} \). Consequently, the maximal k-ideals of \( \mathbb{N} \) are given by \((p)\) for each prime number \( p \).

**Proof.** Obviously, each ideal \((a)\) of \( \mathbb{N} \) is a k-ideal. Now assume that \( A \neq (0) \) is a k-ideal of \( \mathbb{N} \). Let \( a \) be the smallest positive integer contained in \( A \), and \( b \) any element of \( A \). Then \( b = qa + r \) holds for some \( q \in \mathbb{N} \) and \( r \in \mathbb{N} \) satisfying \( 0 \leq r < a \). Since \( r \) belongs to the k-ideal \( A \), it follows that \( r = 0 \), and, hence, \( A = (a) \). The last statement follows since \((a) \subseteq (b)\) holds iff \( b \) divides \( a \).
Remark 4.2. None of the maximal $k$-ideals $(p)$ of $\mathbb{N}$ is a maximal ideal of $\mathbb{N}$. This follows since each ideal $A = (p)$ is properly contained in the proper ideal $B = \{b \in \mathbb{N} | b \geq p\}$ of $\mathbb{N}$.

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