THE BANACH-MAZUR GAME AND GENERIC EXISTENCE
OF SOLUTIONS TO OPTIMIZATION PROBLEMS

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Abstract. The existence of a winning strategy in the well-known Banach-
Mazur game in a completely regular topological space $X$ is proved to be equiv-
alent to the generic existence of solutions of optimization problems generated
by continuous functions in $X$.

1. Introduction

Let $X$ be a completely regular topological space and $C(X)$ denote the space
of all continuous and bounded real-valued functions in $X$. Equipping $C(X)$
with the usual sup-norm $\|f\| := \sup\{|f(x)|: x \in X\}$, under which $C(X)$ is
a Banach space, the following question makes sense: Under what conditions
(necessary and sufficient) on $X$ does the set $E := \{f \in C(X): f$ attains its
minimum in $X\}$ contain a dense and $G_\delta$-subset of $C(X)$? That is, under what
assumptions on $X$ is the set $E$ residual in $C(X)$? Call this property “generic
existence of solutions” of the minimization problems generated by the functions
from $C(X)$.

It turns out that the generic existence of the solutions to the minimization
problems for functions from $C(X)$ is related to the following topological game
in $X$. Two players, named $\alpha$ and $\beta$, play a game in $X$ in the following way:
$\beta$ chooses first a nonempty open subset $U_1$ of $X$. Then $\alpha$ chooses a nonempty
open subset $V_1$ with $V_1 \subset U_1$. Further, $\beta$ chooses a nonempty open subset
$U_2$ of $X$ with $U_2 \subset V_1$ and $\alpha$ chooses a nonempty open $V_2 \subset U_2$ and so on.
The so-obtained infinite sequence $U_1, V_1, \ldots$ is called a play. The player $\alpha$
wins this play if $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$. Otherwise $\beta$ wins.

This game is one of the most known modifications of the Banach-Mazur
game and is denoted usually by $BM(X)$. For the terminology and facts that
we use the reader is referred to the survey [Tel]. Under a strategy for the player
$\alpha$ in the game $BM(X)$ we understand a mapping $s$ that assigns to every chain
$(U_1, V_1, \ldots, U_n)$ corresponding to the first legal $n$ moves of $\beta$ and the first
$n - 1$ moves of $\alpha$, $n \geq 1$, a nonempty open set $V_n \subset U_n$. The strategy $s$
is called winning strategy for the player $\alpha$ (or $\alpha$-winning strategy) if for every
infinite sequence of open sets \( U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots \) such that \( V_n = s(U_1, V_1, \ldots, U_n) \) for every \( n \geq 1 \), we have \( \bigcap_{n=1}^{\infty} V_n \neq \emptyset \). A stationary winning strategy (called also \( \alpha \)-winning tactic (see [Ch])) for the player \( \alpha \) in the game \( BM(X) \) is a strategy for \( \alpha \) that on each step depends only on the last move of the player \( \beta \). Precisely, a stationary winning strategy \( t \) for the player \( \alpha \) is a mapping from the family of nonempty open subsets of \( X \) into the family of nonempty open subsets of \( X \) such that for every nonempty open \( U \subset X \) one has \( t(U) \subset U \) and, moreover, whenever one has a sequence \( (U_n)_{n \geq 1} \) such that \( U_{n+1} \subset t(U_n) \) for every \( n \), then \( \bigcap_{n=1}^{\infty} U_n \neq \emptyset \). Evidently, every \( \alpha \)-winning tactic \( t \) determines the winning strategy \( s(U_1, V_1, \ldots, U_n) := t(U_n) \). There are, however, spaces \( X \) (see [De]) with a winning strategy for the player \( \alpha \) that do not admit an \( \alpha \)-winning tactic. Every space \( X \) that admits a winning strategy for the player \( \alpha \) in the game \( BM(X) \) is a Baire space.

It was proved in [St] that if \( X \) possesses an \( \alpha \)-winning tactic, then one has generic existence of the solution to the minimization problems generated by the functions from \( C(X) \). We show here that this result can be obtained under weaker assumptions on \( X \). It suffices to suppose that \( X \) admits only a winning strategy for the player \( \alpha \) in the game \( BM(X) \) in order to have generic existence of solutions. Moreover, Theorem 3.1 asserts that the generic existence of solutions of the minimization problems determined by functions from \( C(X) \) is a characterization of the fact that \( X \) admits a winning strategy for the player \( \alpha \) in \( BM(X) \). As a corollary we give also another characterization of the spaces \( X \) that possesses an \( \alpha \)-winning strategy (\( \alpha \)-winning tactic) in the case when \( X \) has a \( \sigma \)-discrete net.

2. Some preliminaries

Throughout this article only completely regular topological spaces will be considered. For a subset \( A \) of the topological space \( X \) we denote by \( \text{Int}_X(A) \) and \( \text{Cl}_X(A) \) the interior and the closure of \( A \) in \( X \). If there is no danger of confusion we will write simply \( \text{Int}(A) \) and \( \text{Cl}(A) \) correspondingly.

For a function \( f \in C(X) \) denote by \( M(f) \) the set (possibly empty) of the minimizers of \( f \) in \( X \); i.e.,

\[
M(f) := \{ x \in X : f(x) = \inf \{ f(y) : y \in X \} =: \inf(X, f) \}.
\]

Hence \( M \) is a multivalued mapping from \( C(X) \) onto \( X \) that may have empty values. It can be seen that the domain of \( M \), that is the set \( \text{Dom}(M) := \{ f \in C(X) : M(f) \neq \emptyset \} \), is dense in \( C(X) \). Indeed, if \( f \in C(X) \) and \( \varepsilon > 0 \) are arbitrary, then obviously \( M(f_{\varepsilon}) \neq \emptyset \), where \( f_{\varepsilon}(x) = \max \{ f(x) , \inf(X, f) + \varepsilon \} \). However, the set \( \text{Dom}(M) \) is not obliged to contain a dense and \( G_\delta \)-subset of \( C(X) \).

For a subset \( U \) of \( X \) put \( M^u(U) := \{ f \in C(X) : M(f) \subset U \} \) and for \( W \subset C(X) \) let \( M(W) := \bigcup \{ M(f) : f \in W \} \). Further, given arbitrary \( f \in C(X) \) and \( \varepsilon > 0 \), denote by \( \Omega_f (\varepsilon) \) the set \( \{ x \in X : f(x) < \inf(X, f) + \varepsilon \} \). The sets \( \Omega_f (\varepsilon) \) are nonempty and open for every \( \varepsilon > 0 \). Obviously \( \Omega_f (\varepsilon_1) \subset \Omega_f (\varepsilon_2) \) provided \( \varepsilon_1 \leq \varepsilon_2 \) and, moreover, \( M(f) = \bigcap_{\varepsilon > 0} \Omega_f (\varepsilon) \).

The following proposition summarizes some properties of the mapping \( M \) that we will use later.
Proposition 2.1. The mapping $M$ has the following properties:

(a) $M$ is open, i.e., $M(W)$ is (nonempty) and open in $X$ provided $W$ is (nonempty) and open in $C(X)$;

(b) $\text{Int} M^*(U) \neq \emptyset$ for every nonempty open $U$ in $X$;

(c) for every two open sets $W \subset C(X)$ and $U \subset X$, respectively, with $M(W) \cap U \neq \emptyset$ there exists a nonempty open $W' \subset W$ such that $M(W') \subset U$;

(d) let $\{f_0\} = f_0 \bigcap_{\mathbb{N}} B_n$, where $\{B_n\}_{n \geq 1}$ is a decreasing sequence of subsets in $C(X)$ with $\lim \text{diam}(B_n) = 0$. Then $M(f_0) = \bigcap_{n=1}^\infty M(B_n)$.

Proof. (a) Let $W$ be an open subset of $C(X)$ and $x_0 \in M(f_0)$ for some $f_0 \in W$. Take $\varepsilon > 0$ such that the ball $B(f_0, \varepsilon) := \{f \in C(X) : \|f - f_0\| < \varepsilon\} \subset W$. Then each $x \in B(f_0, \varepsilon)$ is a minimizer of some $f \in W$, e.g., of the function $f_0$ considered above.

(b) Let $x_0 \in U$. Since $X$ is completely regular, there exists $h_0 \in C(X)$ with $h_0(x_0) = 0$, $h_0(X \setminus U) = 1$, and $\|h\| \leq 1$. It is easy to see that $M(B(h_0, 1/3)) \subset U$.

(c) Let $x_0 \in M(f_0) \cap U$ for some $f_0 \in W$. Consider the function $h_0$ from (b) and find $\delta > 0$ such that $f_0 \oplus \delta h_0 \in W$. Let further, $W' \subset W$ be an open set in $C(X)$ containing $f_0 \oplus \delta h_0$ and such that $\text{diam}(W') < \delta/3$. Take $f \in W'$. Since for $x \in X \setminus U$ one has $f(x) \geq (f_0 + \delta h_0)(x) - \delta/3 = f_0(x) + (2\delta)/3 \geq (f_0(x_0) + (2\delta)/3 = (f_0 + \delta h_0)(x_0) + (2\delta)/3 > (f_0 + \delta h_0)(x_0) + \delta/3 \geq f(x_0)$, we see that $M(f) \subset U$.

(d) Obviously $M(f_0) \subset \bigcap_{n=1}^\infty M(B_n)$. On the other hand, take some $x \in M(B_n)$. Then $x \in M(f_n)$ for some $f_n \in B_n$ with $\|f_n - f_0\| \leq \text{diam}(B_n)$. Hence, $x \in \Omega_{f_0}(2\text{diam}(B_n))$. Therefore, $M(B_n) \subset \Omega_{f_0}(2\text{diam}(B_n))$. This gives $\bigcap_{n=1}^\infty M(B_n) \subset \bigcap_{n=1}^\infty \Omega_{f_0}(2\text{diam}(B_n)) = M(f_0)$.

3. The main result

The main result in this article is the following characterization of the fact that the space $X$ possesses an $\alpha$-winning strategy.

Theorem 3.1. The space $X$ admits a winning strategy for the player $\alpha$ in the Banach-Mazur game $BM(X)$ if and only if $\text{Dom}(M)$ contains a dense and $G_\delta$-subset of $C(X)$.

Proof. Sufficiency. Suppose $\text{Dom}(M)$ contains a dense and $G_\delta$-subset of $C(X)$. Then there exist countably many open and dense subsets $(G_n)_{n \geq 1}$ of $C(X)$ such that $\bigcap_{n=1}^\infty G_n \subset \text{Dom}(M)$. The sets $F_n := C(X) \setminus G_n$, $n \geq 1$, are closed and nowhere dense in $C(X)$. That is $\text{Int}(F_n) = \emptyset$ for every $n \geq 1$.

We show that the player $\alpha$ has a winning strategy $s$ in the game $BM(X)$. Let $U_1$ be a nonempty open subset of $X$. Consider the set $\text{Int} M^*(U_1)$ that is nonempty by Proposition 2.1(b). Since $F_1$ is closed and nowhere dense in $C(X)$, the set $\text{Int} M^*(U_1) \setminus F_1$ is nonempty and open in $C(X)$. Take an open ball $B_1$ in $C(X)$ with radius less or equal to 1, such that $B_1 \subset \text{Int} M^*(U_1) \setminus F_1$. Define now the value of the strategy $s$ at $U_1$ by $s(U_1) := M(B_1)$. By Proposition 2.1(a), $s(U_1)$ is a nonempty open subset of $U_1$.

Further, let $U_2$ be an arbitrary nonempty open subset of $V_1 = s(U_1) = M(B_1)$. Since $U_2 \subset M(B_1)$ there is some $f \in B_1$ such that $M(f) \cap U_2 \neq \emptyset$. Hence, by Proposition 2.1(c) there exists a nonempty open $W \subset B_1$ such
that $M(W) \subset U_2$. As above the set $W \setminus F_2$ is a nonempty and open subset of $C(X)$. Take an open ball $B_2$ with radius less or equal to 1/2 such that $\overline{C(B_2)} \subset W \setminus F_2 \subset B_1$ and put $s(U_1, V_1, U_2) := M(B_2)$. Obviously $s(U_1, V_1, U_2)$ is a nonempty open subset of $U_2$. Proceeding by induction we define $s$ for every chain $(U_1, V_1, \ldots, U_n)$, $n \geq 1$, such that $U_k \subset V_{k-1}$ and $V_{k-1} = s(U_1, V_1, \ldots, U_{k-1})$ for every $k$, $2 \leq k \leq n$.

Let $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ be an infinite sequence of open sets in $X$ such that for every $n \geq 1$, $V_n = s(U_1, V_1, \ldots, U_n)$. Let $(B_n)_{n \geq 1}$ be the sequence of open balls in $C(X)$ associated with $(U_n)_{n \geq 1}$ and $(V_n)_{n \geq 1}$ from the construction of $s$. Then for every $n \geq 1$:

1. $\overline{C(B_n + 1)} \subset B_n$ and $B_n \cap F_n = \emptyset$;
2. $\text{diam}(B_n) < 1/n$;
3. $V_n = M(B_n)$.

Conditions (1) and (2) guarantee that $\bigcap_{n=1}^{\infty} B_n$ is a one-point set in $C(X)$, say $f_0$. Moreover, (1) shows in addition that $f_0 \in C(X) \setminus \bigcup_{n=1}^{\infty} F_n \subset \text{Dom}(M)$.

Therefore, by (3) and Proposition 2.1(d) we have

$$\emptyset \neq M(f_0) = \bigcap_{n=1}^{\infty} M(B_n) = \bigcap_{n=1}^{\infty} V_n.$$ 

Hence $s$ is a winning strategy for the player $\alpha$ in the Banach-Mazur game $BM(X)$.

**Necessity.** Suppose now, that there exists a winning strategy $s$ for the player $\alpha$ in the game $BM(X)$. Every finite sequence of sets $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, obtained by the first $n$ steps in the game $BM(X)$ is called a partial play in this game. The key step in the proof is the following lemma:

**Lemma 3.2.** Let $(U_1, V_1, \ldots, U_n, V_n)$, $n \geq 1$, be a partial play in the game $BM(X)$ and $W_n$ be a nonempty open subset of $C(X)$ such that $M(W_n) \subset V_n$. Then there is a family $\Gamma(W_n)$ of triples $(U_{n+1}, V_{n+1}, W_{n+1})$ such that:

1. $U_{n+1}$ is a nonempty open subset of $M(W_n)$;
2. $V_{n+1} = s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$;
3. $W_{n+1}$ is a nonempty subset of $C(X)$ such that $\text{diam}(W_{n+1}) < 1/(n + 1)$, $\overline{C(W_{n+1})} \subset W_n$, and $M(W_{n+1}) \subset V_{n+1}$;
4. the family $\gamma(W_n) := \{W_{n+1} : (U_{n+1}, V_{n+1}, W_{n+1}) \in \Gamma(W_n) \text{ for some } U_{n+1}, V_{n+1} \}$ is disjoint;
5. the set $H(W_n) := \bigcup\{W_{n+1} : W_{n+1} \in \gamma(W_n)\}$ is dense in $W_n$.

**Proof of Lemma 3.2.** Take a maximal family $\Gamma(W_n)$ satisfying the properties (a)–(d). We prove that it satisfies also the condition (e).

Suppose the contrary. There exists a nonempty open subset $G$ of $C(X)$ with $G \subset W_n$ and $G \cap H(W_n) = \emptyset$. Consider the set $M(G)$ that, by Proposition 2.1(a), is a nonempty and open subset of $X$. Moreover, $M(G) \subset M(W_n) \subset V_n$. Let $U_{n+1} := M(G)$ and $V_{n+1} := s(U_1, V_1, \ldots, U_n, V_n, U_{n+1})$. By Proposition 2.1(c) there is a nonempty open subset $W_{n+1}$ of $C(X)$ such that $W_{n+1} \subset G$ and $M(W_{n+1}) \subset V_{n+1}$. We may arrange, in addition, $\overline{C(W_{n+1})} \subset W_n$ and $\text{diam}(W_{n+1}) < 1/(n + 1)$. Now, the family $\Gamma' := \Gamma(W_n) \cup \{(U_{n+1}, V_{n+1}, W_{n+1})\}$ is strictly larger than $\Gamma(W_n)$ and satisfies (a)–(d). This is a contradiction showing that the maximal family $\Gamma(W_n)$ satisfies also (e). □
Let us mention that Lemma 3.2 is true also for \( n = 0 \) provided we put \( U_0 = V_0 = X \). Now, let us get back to the proof of the theorem. We proceed in the following way.

Put \( \gamma_0 := \{C(X)\}, \quad W_0 = C(X), \quad U_0 = V_0 = X \) and apply Lemma 3.2 for the triple \((U_0, V_0, W_0)\). We get a family of triples \( \Gamma_1 := \Gamma_1(W_0) \) satisfying conditions (a)-(e) from Lemma 3.2. Put \( \gamma_1 := \gamma(W_0) \) and \( H_1 := H(W_0) \). By (e) the set \( H_1 \) is open and dense in \( C(X) \). Further, because of (d), for every \( W_1 \in \gamma_1 \) there is a unique couple \((U_1, V_1)\) with \((U_1, V_1, W_1) \in \Gamma_1 \). Apply again Lemma 3.2 for this triple. As a result, for every \( W_1 \in \gamma_1 \) we obtain a family of triples \( \Gamma(W_1) \) with the properties (a)-(e) fulfilled with respect to the couple \((U_1, V_1)\) corresponding to \( W_1 \). Let \( \gamma_2 := \bigcup\{\gamma(W_1) : W_1 \in \gamma_1\} \), \( \gamma_2 := \bigcup\{\gamma(W_1) : W_1 \in \gamma_1\} \), and \( H_2 := \bigcup\{H(W_1) : W_1 \in \gamma_1\} \). Since \( \gamma_1 \) is disjoint and each \( \gamma(W_1) \) is disjoint too, then the family \( \gamma_2 \) is also disjoint. Moreover, by (e) every \( H(W_1) \) is dense in \( W_1 \) and since \( H_1 \) is dense in \( C(X) \), it follows that \( H_2 \) is (open) and dense in \( C(X) \) as well.

Proceeding in this way we obtain a sequence of families \((\Gamma_n)_{n \geq 1}\) of triples and a sequence of disjoint families \((\gamma_n)_{n \geq 0}\) of open sets in \( C(X) \), with \( \gamma_0 = \{C(X)\} \), such that for every \( n \geq 1 \) we have

(i) \( \Gamma_n \) is a union of the families \( \Gamma(W_{n-1}) \), \( W_{n-1} \in \gamma_{n-1} \), where \( \Gamma(W_{n-1}) \) is obtained by Lemma 3.2 from some uniquely determined partial play \((U_1, V_1, \ldots, U_{n-1}, V_{n-1})\);

(ii) \( \gamma_n \) is a union of the families \( \gamma(W_{n-1}) \), \( W_{n-1} \in \gamma_{n-1} \) from the condition (d) of Lemma 3.2;

(iii) the set \( H_n := \bigcup\{W_n : W_n \in \gamma_n\} \) is open and dense in \( C(X) \).

Let \( H_0 := \bigcap_{n=1}^{\infty} H_n \). Obviously \( H_0 \) is a dense and \( G_\delta \)-subset of \( C(X) \). Take \( f_0 \in H_0 \). By the properties above, this \( f_0 \) determines a unique sequence \((W_n)_{n \geq 1}\) such that for every \( n \geq 1 \), \( W_n \in \gamma_n \), \( \text{Cl}(W_{n+1}) \subset W_n \), and \( \text{diam}(W_n) < 1/n \). Hence \( \{f_0\} = \bigcap_{n=1}^{\infty} W_n \). By the properties (a)-(d) from Lemma 3.2 and conditions (i)-(iii) above it follows that there is an infinite sequence of open sets

\[
U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots
\]

such that \( V_n = s(U_1, V_1, \ldots, U_n) \) and \( U_{n+1} \subset M(W_n) \subset V_n \) for every \( n \geq 1 \).

Hence, by Proposition 2.1(d) we have

\[
M(f_0) = \bigcap_{n=1}^{\infty} M(W_n) = \bigcap_{n=1}^{\infty} V_n = \bigcap_{n=1}^{\infty} U_n.
\]

Since \( s \) is a winning strategy, we see that \( M(f_0) = \bigcap_{n=1}^{\infty} V_n \neq \emptyset \). The proof is complete.

As an immediate corollary from Theorem 3.1 we get the following sufficient condition for the set \( \text{Dom}(M) \) to be residual in \( C(X) \).

**Corollary 3.3** (see [St, Theorem 5]). Let \( X \) admit an \( \alpha \)-winning tactic in the Banach-Mazur game. Then the set \( \text{Dom}(M) \) contains a dense and \( G_\delta \)-subset of \( C(X) \).

As mentioned above, there exists a completely regular space \( X \) with \( \alpha \)-winning strategy that does not admit an \( \alpha \)-winning tactic (see [De]).
A slight change in the proof of Theorem 3.1 gives us the possibility to characterize the spaces $X$ for which the set $\{f \in C(X) : f \text{ attains its minimum in } X \text{ at exactly one point}\}$ contains a dense and $G_\delta$-subset of $C(X)$, i.e., to characterize the spaces $X$ in which we have generic uniqueness of the solution of the minimization problems generated by functions from $C(X)$.

**Theorem 3.4.** The set $\{f \in C(X) : f \text{ attains its minimum in } X \text{ at exactly one point}\}$ contains a dense and $G_\delta$-subset of $C(X)$ if and only if the space $X$ admits an $\alpha$-winning strategy $s$ such that, whenever one has a sequence of open sets $U_1 \supset V_1 \supset \cdots \supset U_n \supset V_n \supset \cdots$ with $V_n = s(U_1, V_1, \ldots, U_n)$ for every $n \geq 1$, then $\bigcap_{n=1}^\infty V_n$ is a one-point set.

Let us mention a class of spaces for which we have generic uniqueness of the solution for the minimization problems for the functions from $C(X)$. The topological space $X$ is called fragmentable (see [JaRo]) if there is a metric $\rho$ on it such that for every $\varepsilon > 0$ and every nonempty subset $Y$ of $X$ there exists a nonempty relatively open subset $A$ of $Y$ with $\rho$-diam$(A) < \varepsilon$. Further information about fragmentable spaces can be found in [Na, Ri].

**Corollary 3.5.** Let $X$ be a fragmentable space that admits an $\alpha$-winning strategy. Then the set $\{f \in C(X) : f \text{ attains its minimum in } X \text{ at exactly one point}\}$ contains a dense and $G_\delta$-subset of $C(X)$.

**Proof.** Let $\rho$ be the metric on $X$ that fragments it. Let $U_1, V_1, \ldots, U_n$ be the first $n$ steps of the player $\beta$ and the first $n-1$ steps of $\alpha$ in the game $BM(X)$, $n \geq 1$. Take a nonempty open set $U'_n \subset U_n$ such that $\rho$-diam$(U'_n) < 1/n$, and define $s'(U_1, V_1, \ldots, U'_n) = s(U_1, V_1, \ldots, U_n)$. It is easy to check that the so obtained strategy $s'$ satisfies the requirements of Theorem 3.4. $\square$

We give some further corollaries of Theorem 3.1. Before that let us recall some notions. A minimization problem generated by some $f \in C(X)$ (which we will denote by $(X, f)$) is said to be Tikhonov well posed (see [Ti]) if it has unique solution $x_0 \in X$ and, moreover, every minimizing sequence $(x_n)_{n \geq 1}$ (i.e., $f(x_n) \to \inf(X, f)$) converges to this unique solution. If $(X, f)$ is Tikhonov well posed then every minimizing net (not only every minimizing sequence) converges to its unique solution (see, e.g., [ČKR]). Let

$$T := \{f \in C(X) : (X, f) \text{ is Tikhonov well posed}\}.$$ 

The following fact is proved for compact spaces $X$ in [ČK1, ČK2] and for an arbitrary $X$ in [ČKR, Theorem 3.5]: The set $T$ contains a dense and $G_\delta$-subset of $C(X)$ iff the space $X$ contains a dense and completely metrizable subspace.

A family $\mathcal{F}$ of subsets of $X$ is called a net in $X$ if for every $x \in X$ and every open $U \subset X$, with $x \in U$, there exists $H \in \mathcal{F}$ such that $x \in H \subset U$. The space $X$ possesses a $\sigma$-discrete net if there are countably many discrete families $(\gamma_n)_{n \geq 1}$ in $X$ such that $\gamma := \bigcup_{n=1}^\infty \gamma_n$ forms a net in $X$. Recall that a family of subsets of $X$ is discrete if every point in $X$ has a neighborhood that intersects at most one element of the family. Every metric space has a $\sigma$-discrete net. The following is again a result proved in [ČKR, Theorem 5.6]: Suppose that $X$ possesses a $\sigma$-discrete net. Then the set $T \cup (C(X) \setminus \text{Dom}(M))$ contains a dense and $G_\delta$-subset of $C(X)$. 

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In the special case when $X$ has a $\sigma$-discrete net we can give another characterization of the fact that the space $X$ admits an $\alpha$-winning strategy. For a metric space $X$ this characterization follows by a result of [Ox] and for a space $X$ with a base of countable order can be found in [Wh].

**Theorem 3.6.** Let $X$ possess a $\sigma$-discrete net. Then $X$ admits an $\alpha$-winning strategy in the game $BM(X)$ if and only if $X$ contains a dense and completely metrizable subspace.

**Proof.** The proof is an immediate consequence from Theorem 3.1 and the mentioned results from [ČKR]. □

Every space $X$ that contains a dense and completely metrizable subspace possesses $\alpha$-winning tactic (see, e.g., [Tel]). Then we have

**Corollary 3.7.** Let $X$ possess a $\sigma$-discrete net. Then $X$ admits $\alpha$-winning strategy in the game $BM(X)$ if and only if it admits $\alpha$-winning tactic in this game.

**References**


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