C*-EXTREME POINTS
OF SOME COMPACT C*-CONVEX SETS

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Abstract. In the C*-algebra $M_n$ of complex $n \times n$ matrices, we consider the notion of noncommutative convexity called C*-convexity and the corresponding notion of a C*-extreme point. We prove that each irreducible element of $M_n$ is a C*-extreme point of the C*-convex set it generates, and we classify the C*-extreme points of any C*-convex set generated by a compact set of normal matrices.

1. Introduction

For C*-algebras and, more generally, for bimodules over C*-algebras there is a notion of convexity that incorporates algebra-valued convex coefficients in a natural way. This form of convexity, called C*-convexity, was studied in its own right by Loebl and Paulsen in [6] and by Hopenwasser, Moore, and Paulsen in [5]. There is an extremal theory associated with C*-convexity and it is our aim in this paper to further its development by studying the C*-extreme points of specific compact C*-convex sets in the C*-algebra $M_n$ of complex $n \times n$ matrices.

Sections 3 and 4 contain the main results. In §3 it is proved that every irreducible element of $M_n$ is a C*-extreme point of the C*-convex set it generates. In §4 conditions are presented that must be satisfied by the C*-extreme points of a C*-convex set generated by a family of matrices; these necessary conditions lead to a classification of the C*-extreme points of compact C*-convex sets generated by normals. Auxiliary results include an extension of Stampfli's theorem concerning the extreme points of the classical numerical range of a hyponormal operator and a result to the effect that irreducible elements $x \in M_n$ can be determined up to unitary equivalence among all elements of $M_n$ by the norms of linear polynomials in $x$ with coefficients taken from $M_n$.

We begin by recalling the definitions [6] of C*-convexity and C*-extreme point. A subset $K$ of a unital C*-algebra $A$ is said to be C*-convex if $K$ is closed under the formation of finite sums of the type $\sum_i t_i^* x_i t_i$, where each
$x_i \in K$ and the elements $t_i \in A$ are under the proviso that $\sum_i t_i^* t_i = 1$. The $t_i$ are called $C^*$-convex coefficients and the $C^*$-convex combination $\sum_i t_i^* x_i t_i$ is called proper if each coefficient $t_i$ is invertible in $A$. An element $x$ in a $C^*$-convex set $K \subset A$ is said to be a $C^*$-extreme point of $K$ if the only possible representations of $x$ as proper $C^*$-convex combinations of elements $x_i \in K$ are those for which each $x_i$ comes from the unitary orbit of $x$. To distinguish extreme points from $C^*$-extreme points, the former are sometimes referred to as linear extreme points.

In [6] it is shown that the $C^*$-extreme points of $C^*$-convex subsets of $M_n$ are linear extreme points, and in [5] an example in $M_2$ of a $C^*$-convex set having some linear extreme points that are not $C^*$-extreme is provided. In §4 we present a new class of examples, occurring in every matrix algebra, that show that linear extreme points need not be $C^*$-extreme. Although a compact $C^*$-convex subset of $M_n$ always possesses $C^*$-extreme points [4], the Krein-Milman type problem of whether such extreme points exist in sufficient numbers to recover the original set remains open.

Complete information about the $C^*$-extreme points of certain $C^*$-convex sets is available in a few special cases: the closed unit ball of the algebra of operators acting on a Hilbert space, the closed interval $\{x : 0 \leq x \leq 1\}$ of operators, and the set of $x \in M_2$ with numerical radius $w(x) \leq 1$ (see [5]). We augment these results by completing the classification begun in [4] of the $C^*$-extreme points of $C^*$-convex sets generated by compact sets of normal matrices (Corollary 4.2) and by determining all of the $C^*$-extreme points of the $C^*$-convex set generated by a single $2 \times 2$ matrix (Theorem 4.3).

One source for interest in $C^*$-convexity is the theory of matricial ranges [1, §2.4; 3; 9, §4]. The $n$th matricial range of an element $a$ in a unital $C^*$-algebra $A$ is the compact $C^*$-convex subset $W^n(a) \subset M_n$ of elements of the form $\phi(a)$, where $\phi$ denotes a unital completely positive map of $A$ into $M_n$. If $x$ is an element of $B(H)$, the $C^*$-algebra of bounded linear operators on a complex Hilbert space $H$, then the numerical range $W^1(x)$ of $x$ is the closure of the classical numerical range $\{(x\xi, \xi) : \xi \in H, ||\xi|| = 1\}$. In §5 some of our results on $C^*$-extreme points are applied to the matricial ranges of hyponormal operators.

The $C^*$-convex hull of a subset $\mathcal{S} \subset M_n$ is the smallest $C^*$-convex set containing $\mathcal{S}$ and is denoted by $C^*$-conv $\mathcal{S}$. An essential fact is that $C^*$-conv $\mathcal{S}$ is compact whenever $\mathcal{S}$ is a compact subset of $M_n$ [3, 2.4].

The unitary equivalence of $x$ and $z$ is denoted by $x \sim z$ and is taken to mean that there is a unitary $u$ such that $x = u^* zu$.

2. ON A COMPLETE UNITARY INVARIANT OF IRREDUCIBLE MATRICES

The (finite) order of an irreducible operator $x \in B(H)$ is the least positive integer $k$, if it exists, for which the values of $\| \sum_{i=0}^k x^i \otimes c_i \|$, where $c_0, c_1, \ldots, c_k$ are complex $n \times n$ matrices with $n$ arbitrary, determine $x$ among the irreducible elements of $B(H)$ uniquely up to unitary equivalence. An important theorem of Arveson asserts that for irreducible compact operators acting on a complex separable Hilbert space, the order exists and is equal to 1 [1, Chapter 2]; in particular, irreducible elements in the $C^*$-algebra $M_n$ are of first order. The purpose of this section is to show that for the algebra $M_n$ the situation is
somewhat more special in that irreducible elements \( x \in M_n \) can be determined up to unitary equivalence among all elements of \( M_n \) by the norms of linear matrix polynomials \( x \otimes c_1 + 1 \otimes c_0 \) in just one algebra only—\( M_n \otimes M_n \).

2.1. **Theorem.** Suppose that \( x, z \in M_n \) satisfy \( \| x \otimes b + 1 \otimes m \| = \| z \otimes b + 1 \otimes m \| \) for every \( b, m \in M_n \). If \( x \) is irreducible, that is, if the only selfadjoint idempotents commuting with \( x \) are 0 and 1, then \( z \) must be irreducible and, moreover, unitarily equivalent to \( x \).

As in Arveson's paper [1], the proof of Theorem 2.1 involves the use of his boundary theorem and the comparison of the matricial ranges of \( z \) with those of \( x \). It is in this latter aspect that the connection with \( C^* \)-convexity theory arises.

Every unital completely positive map \( \phi : M_n \rightarrow M_n \) has the form \( \phi(\cdot) = \sum t_i^* (\cdot) t_i \), where the sum is a finite sum and the matrices \( t_i \) satisfy \( \sum t_i^* t_i = 1 \) [2]. Hence, the \( C^* \)-convex set generated by \( x \in M_n \) is precisely the \( n \)th matricial range of \( x \), and so we are able to make use of some of the results of [3; 9] in a manner summarized by the

2.2. **Theorem.** For \( x, z \in M_n \), the following statements are equivalent.

1. \( C^* \)-conv\{\( x \)\} \( \subseteq \) \( C^* \)-conv\{\( z \)\};
2. \( W^1(x \otimes b + 1 \otimes m) \subseteq W^1(z \otimes b + 1 \otimes m) \) for every \( b, m \in M_n \);
3. \( \| x \otimes b + 1 \otimes m \| \leq \| z \otimes b + 1 \otimes m \| \) for every \( b, m \in M_n \).

**Proof.** In identifying \( C^* \)-conv\{\( x \)\} with \( W^n(x) \) and \( C^* \)-conv\{\( z \)\} with \( W^n(z) \), the implication (3) \( \Rightarrow \) (1) is given by [3, 2.1] and the equivalence of (1) and (2) by [9, 4.3]. To prove the implication (1) \( \Rightarrow \) (3), observe that because \( x \in W^n(x) \subseteq W^n(z) \), there is a unital completely positive \( \phi : M_n \rightarrow M_n \) with \( x = \phi(z) \). The induced map \( \phi \otimes \text{id} \) on \( M_n \otimes M_n \), where \( \text{id} : M_n \rightarrow M_n \) is the identity map, is a contraction, and therefore for each \( b, m \in M_n \),

\[
\| x \otimes b + 1 \otimes m \| = \| \phi(z) \otimes b + 1 \otimes m \| \leq \| z \otimes b + 1 \otimes m \|.
\]

We now provide the

**Proof of Theorem 2.1.** Using Theorem 2.2, the hypothesis is that \( W^n(x) = W^n(z) \). There exist, therefore, unital completely positive maps \( \phi, \psi : M_n \rightarrow M_n \) such that \( x = \psi(z) \) and \( z = \phi(x) \), whence \( x = \psi \circ \phi(x) \). Let \( \mathcal{L} \) denote the subspace spanned by \( x, x^* \), and 1, and let \( \psi \) denote the identity map \( \mathcal{L} \rightarrow M_n \): namely, \( \psi = \psi \circ \phi \mid_{\mathcal{L}} \). Because \( \mathcal{L} \) is irreducible and the \( C^* \)-algebra \( \mathcal{L} \) generates is \( M_n \), the boundary theorem of Arveson [1, 2.1.1] states that \( \psi \) has a unique completely positive extension to \( M_n \). The identity map and \( \psi \circ \phi \), being two such extensions of \( \psi \), must therefore coincide.

From \( \text{id} = \psi \circ \phi \), it follows that, for every \( a \in M_n \),

\[
\| a \| = \| \psi(\phi(a)) \| \leq \| \phi(a) \| \leq \| a \|;
\]

that is, \( \phi \) is a unital isometry on \( M_n \) and, therefore, has the form \( x \mapsto u^* x u \) or \( x \mapsto u^* x^t u \) for some unitary \( u \in M_n \) [7]. Complete positivity rules out the latter possibility, so that \( z = \phi(x) = u^* x u \) as desired. □

3. **\( C^* \)-convex sets with a single, irreducible generator**

Using the main result of §2, we will prove that each irreducible element of \( M_n \) is \( C^* \)-extreme in the \( C^* \)-convex set it generates.
3.1. **Theorem.** If \( x \in M_n \) is irreducible, then \( x \) is a \( C^* \)-extreme point of its \( C^* \)-convex hull.

**Proof.** Suppose that \( x = \sum_i t_i^* x_i t_i \) for some elements \( x_i \in C^*\text{-conv} \{x\} \) and for some invertible \( C^* \)-convex coefficients \( t_i \in M_n \). Because each \( x_i \in C^*\text{-conv} \{x\} \), it follows from (2) of Theorem 2.2 that \( W^1(x_i \otimes b + 1 \otimes m) \subset W^1(x \otimes b + 1 \otimes m) \) for every \( i \) and for every \( b, m \in M_n \). We now show that the reverse inclusions \( W^1(x \otimes b + 1 \otimes m) \subset W^1(x_i \otimes b + 1 \otimes m) \) hold.

Fix \( b, m \in M_n \). Let \( \lambda \) be an arbitrary extreme point of \( W^1(x \otimes b + 1 \otimes m) \); so \( \lambda \) is determined by some unit vector \( \xi \in \mathbb{C}^n \otimes \mathbb{C}^n \). By using the unit vectors

\[
\eta_i = ||(t_i \otimes 1)\xi||^{-1}(t_i \otimes 1)\xi
\]

and the induced elements

\[
\lambda_i = ((x_i \otimes b + 1 \otimes m)\eta_i, \eta_i),
\]

the extreme point \( \lambda \) can be expressed as a (proper) convex combination of the \( \lambda_i \in W^1(x_i \otimes b + 1 \otimes m) \) as follows:

\[
\lambda = ((x \otimes b + 1 \otimes m)\xi, \xi) = \sum_i ((x_i \otimes b + 1 \otimes m)(t_i \otimes 1)\xi, (t_i \otimes 1)\xi) = \sum_i ||(t_i \otimes 1)\xi||^2 \lambda_i.
\]

But each \( \lambda_i \) is an element of \( W^1(x \otimes b + 1 \otimes m) \) and therefore, because \( \lambda \) is an extreme point of \( W^1(x \otimes b + 1 \otimes m) \), we must have \( \lambda_i = \lambda \) for every \( i \). This proves that the extreme points of \( W^1(x \otimes b + 1 \otimes m) \) lie in each of the sets \( W^1(x_i \otimes b + 1 \otimes m) \). Hence, \( W^1(x \otimes b + 1 \otimes m) \) is contained in each \( W^1(x_i \otimes b + 1 \otimes m) \) and so, by Theorem 2.2, \( C^*\text{-conv} \{x_i\} = C^*\text{-conv} \{x\} \) for every \( i \). Equivalently, \( ||x_i \otimes b + 1 \otimes m|| = ||x \otimes b + 1 \otimes m|| \) for every \( b, m \in M_n \) and every \( i \). But \( x \) is irreducible, and so as Theorem 2.1 asserts, each \( x_i \) is unitarily equivalent to \( x \); that is, \( x \) is a \( C^* \)-extreme point of its \( C^* \)-convex hull. \( \square \)

4. **The search for \( C^* \)-extreme points; narrowing down the possibilities**

Before we can hope to answer the Krein-Milman question of whether the \( C^* \)-extreme points are sufficient to recover the original \( C^* \)-convex set, it would help to be able to describe all the \( C^* \)-extreme points. The purpose of this section is to present necessary conditions for points to be \( C^* \)-extreme in various cases in order to narrow the search considerably. The main results are Theorem 4.1 and its applications.

We begin by presenting techniques for rewriting \( C^* \)-convex combinations.

**Technique A.** Let \( x = \sum_{i=1}^m t_i^* x_i t_i \) with \( \sum_i t_i^* t_i = 1 \) and \( ||t_i|| < 1 \) for each \( i \). If this is not already a proper \( C^* \)-convex combination, then we can rewrite it as one in the following manner. Let

\[
a_i = (m - 1)^{-1/2}(1 - t_i^* t_i)^{1/2}
\]

(invertible since \( ||t_i|| < 1 \)),

\[
\kappa_{ij} = \frac{1 - \delta_{ij}}{m - 1}, \quad h_{ij} = \sqrt{k_{ij} t_j a_i^{-1}}.
\]
For each \( i \),
\[
\sum_j h_{ij}^* h_{ij} = \sum_j \sqrt{\kappa_{ij}} (a_i^{-1})^* t_i^* t_i a_i^{-1} \sqrt{\kappa_{ij}}
\]
\[
= (1 - t_i^* t_i)^{-1/2} \left( \sum_j (1 - \delta_{ij}) t_j^* t_j \right) (1 - t_i^* t_i)^{-1/2}
\]
\[
= (1 - t_i^* t_i)^{-1/2} (1 - t_i^* t_i) (1 - t_i^* t_i)^{-1/2} = 1,
\]
and so \( z_i = \sum_{j=1}^m h_{ij}^* x_j h_{ij} \in C^*\text{-conv}\{x_1, \ldots, x_m\} \) by the above computation.
Further, \( x = \sum_i a_i^* z_i a_i \) is a representation of \( x \) as a proper \( C^* \)-convex combination of the \( z_i \). An important subtlety that will be essential later is that each \( h_{ij} = 0 \) and so each \( z_i \) is a \( C^* \)-convex combination of the \( x_j \) using fewer than \( m \) nonzero coefficients.

**Technique B.** If \( p_i^* p_i + p_2^* p_2 = 1 \) and if
\[
*-(\sigma > \sigma) + (1 - p_i^* p_i) > (1 - p_2^* p_2)\]
then for every scalar \( \lambda \in (0, 1) \), \( x \) can be rewritten as
\[
x = \begin{pmatrix} 1 & 0 \\ 0 & p_1^* \end{pmatrix} x_1 \begin{pmatrix} 1 & 0 \\ 0 & p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p_2^* \end{pmatrix} x_2 \begin{pmatrix} 0 & 0 \\ 0 & p_2 \end{pmatrix},
\]
then for every scalar \( \lambda \in (0, 1) \), \( x \) can be rewritten as
\[
x = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda p_1^* \end{pmatrix} x_1 \begin{pmatrix} \lambda & 0 \\ 0 & \lambda p_1 \end{pmatrix} + \begin{pmatrix} \sqrt{1 - \lambda^2} & 0 \\ 0 & (1 - \lambda^2 p_1^* p_1)^{1/2} \end{pmatrix} x' \begin{pmatrix} \sqrt{1 - \lambda^2} & 0 \\ 0 & (1 - \lambda^2 p_1^* p_1)^{1/2} \end{pmatrix},
\]
where
\[
x' = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \lambda^2} (1 - \lambda^2 p_1^* p_1)^{-1/2} p_1^* \end{pmatrix} x_1 \begin{pmatrix} 1 & 0 \\ 0 & p_1 \sqrt{1 - \lambda^2} (1 - \lambda^2 p_1^* p_1)^{-1/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (1 - \lambda^2 p_1^* p_1)^{-1/2} p_2^* \end{pmatrix} x_2 \begin{pmatrix} 0 & 0 \\ 0 & p_2 (1 - \lambda^2 p_1^* p_1)^{-1/2} \end{pmatrix}.
\]
Observe that in order for \( x \) to be a proper \( C^* \)-convex combination of \( x_1 \) and \( x' \), it is sufficient that \( p_1 \) be invertible.

**Technique C.** If \( \sum_{i=1}^m p_i^* p_i \) is invertible, then
\[
\sum_i \begin{pmatrix} 0 & 0 \\ 0 & p_i^* \end{pmatrix} x_i \begin{pmatrix} 0 & 0 \\ 0 & p_i \end{pmatrix}
\]
can be rewritten as
\[
\begin{pmatrix} 0 \\ 0 \end{pmatrix} (\sum_j p_j^* p_j)^{1/2} z \begin{pmatrix} 0 \\ 0 \end{pmatrix} (\sum_j p_j^* p_j)^{1/2},
\]
where \( z \in C^*\text{-conv}\{x_i\} \) and is given by
\[
z = \sum_i \begin{pmatrix} m^{-1/2} & 0 \\ 0 & (\sum_j p_j^* p_j)^{-1/2} p_i^* \end{pmatrix} x_i \begin{pmatrix} m^{-1/2} & 0 \\ 0 & p_i (\sum_j p_j^* p_j)^{-1/2} \end{pmatrix}.
\]

Finally, we will make use of the following two elementary facts. One is that if \( K \subseteq M_n \) is \( C^* \)-convex, and if \( p \in M_n \) is a projection, then the compression \( p K p \) of \( K \) to the image of \( p \) is clearly \( C^* \)-convex. The other is that \( x = x_1 \oplus x_2 \).
is $C^*$-extreme in $K$ only if $x_1$ and $x_2$ are $C^*$-extreme in the corresponding compressed images of $K$. To prove the second fact, suppose that $x_1$, say, is not $C^*$-extreme in $K_1$, the corresponding compression of $K$. Then $x_1$ can be expressed as a proper $C^*$-convex combination $\sum_{i=1}^{m} s_i^* z_i s_i$ of elements $z_i \in K_1$, with at least one $z_i \not\sim x_1$. Thus,

$$x = \sum_{i=1}^{m} (s_i \oplus m^{-1/2} 1)^* (z_i \oplus x_2) (s_i \oplus m^{-1/2} 1)$$

is a representation of $x$ as a proper $C^*$-convex combination of elements $z_i \oplus x_2 \in K$, where at least one $z_i \oplus x_2 \not\sim x_1 \oplus x_2$.

4.1. **Theorem.** Suppose that $K = C^*$-conv{$x_\alpha : \alpha \in I$} $\subset M_n$, where $I$ is any index set. If $x$ is $C^*$-extreme in $K$, then $x$ must be unitary equivalent to some $x_\alpha$ or reducible. Moreover, there exist projections $q_i$ such that $\sum_i q_i = 1$, $x = \sum_i q_i x'_i q_i$, and each $x'_i \sim x_\alpha$.

**Proof.** Let $x = \sum_{i=1}^{m} t_i x_\alpha t_i$ be a representation of $x$ as a $C^*$-convex combination of the $x_\alpha$ using the fewest number of coefficients possible. (This least integer $m$ does not change if $x$ is replaced by some $x' \sim x$.) Because $m = 1$ if and only if $x \sim x_{\alpha_1}$, it is assumed henceforth that $m \geq 2$ and that $x$ is not unitarily equivalent to some $x_\alpha$. We are to prove that $x$ is reducible.

We will assume without loss of generality that $\|t_i\| = 1$. Such an assumption is valid for the following reasons. If, on the contrary, $\|t_i\| < 1$ for every $i$, then $x$ can be rewritten using Technique A as a proper $C^*$-convex combination of elements $z_i$. Because $x$ is a $C^*$-extreme point of $K$, each $z_i$ must be unitarily equivalent to $x$. However, the construction in Technique A shows that each $z_i$ is a $C^*$-convex combination of fewer than $m$ of the $x_\alpha$; hence, $x$ must possess this property of the $z_i$ as well, in contradiction of the minimality of $m$. Thus, at least one coefficient $t_i$ has unit norm.

By expressing each coefficient $t_i$ in its polar decomposition $t_i = u_i a_i$, where $u_i$ is unitary and $a_i \geq 0$, and by absorbing the unitary part $u_i$ of $t_i$ into $x_{\alpha_i}$, we have that $x = \sum_i a_i x'_i a_i$, where $a_i \geq 0$, $\sum_i a_i^2 = 1$, and $x'_i = u_i^* x_{\alpha_i} u_i$. There is a unitary $u \in M_n$ for which $u^* a_i u$ has a representation as a $2 \times 2$ operator matrix given by

$$u^* a_i u = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix}$$

with $y_1 \geq 0$ and $\|y_1\| < 1$. (This follows from the fact that $a_i \geq 0$ and $\|a_i\| = 1$.) Because $a_i \geq 0$ for all $i \geq 2$ and because $\sum_i a_i^2 = 1$, the same unitary $u$ gives

$$u^* a_i u = \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix}$$

for $i \geq 2$.

Again, each $y_i \geq 0$. Hence,

$$x' = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} x''_{\alpha_1} \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} + \sum_{i \geq 2} \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix} x''_{\alpha_i} \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix},$$

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where \( x' = u x u^* \) and \( x'' = u^* x' u \); for simplicity of notation, let us assume that
\[
\begin{align*}
  a_1 &= \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix}, & a_i &= \begin{pmatrix} 0 & 0 \\ 0 & y_i \end{pmatrix},
\end{align*}
\]
et cetera, so that \( x = \sum_i a_i x_{\alpha_i} a_i \).

Now because \( \|y_1\| < 1, \ 1 - y_1^2 \) is invertible; therefore, from
\[
\sum_{i \geq 2} a_i^2 = \sum_{i \geq 2} \begin{pmatrix} 0 & 0 \\ 0 & y_i^2 \end{pmatrix} = 1 - a_1^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 - y_1^2 \end{pmatrix},
\]
we conclude that \( \sum_{i \geq 2} y_i y_i^* \) is invertible. Hence, Technique C allows us to write \( \sum_{i \geq 2} a_i x_{\alpha_i} a_i \) as a single term \( t_0^* x_0 a_0 \), for some \( x_0 \in K \). By passing to the polar decomposition of \( t_0 \) and absorbing the unitary part of \( t_0 \) into \( x_0 \), we may assume that \( \sum_{i \geq 2} a_i x_{\alpha_i} a_i = a_0 x_0 a_0 \) for some \( a_0 \geq 0 \) and \( x_0 \in K \).

From \( a_0^2 + a_1^2 = 1 \), we see that
\[
a_0 = \begin{pmatrix} 0 & 0 \\ 0 & y_0 \end{pmatrix}.
\]

In fact, \( a_0 \) and \( a_1 \) admit a further decomposition so that \( x \) is given by
\[
x = \begin{pmatrix} 1 & p_1 \\ 0 & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} 1 & p_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & p_0 \\ 0 & 1 \end{pmatrix} x_0 \begin{pmatrix} 0 & p_0 \\ 0 & 1 \end{pmatrix},
\]
where \( p_1 \geq 0 \) is invertible and is of norm \( \|p_1\| < 1 \). We are now in a position to make repeated use of Technique B.

For each \( \lambda \in (0, 1) \), we rewrite \( x \) (using Technique B) as
\[
x = \begin{pmatrix} \lambda & \lambda p_1 \\ \lambda p_1 & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} \lambda & \lambda p_1 \\ \lambda p_1 & 0 \end{pmatrix} + \begin{pmatrix} \sqrt{1 - \lambda^2} & (1 - \lambda^2 p_1^2)^{1/2} \\ (1 - \lambda^2 p_1^2)^{1/2} & 1 \end{pmatrix} x(\lambda) \begin{pmatrix} \sqrt{1 - \lambda^2} & (1 - \lambda^2 p_1^2)^{1/2} \\ (1 - \lambda^2 p_1^2)^{1/2} & 1 \end{pmatrix},
\]
where
\[
x(\lambda) = \begin{pmatrix} 1 & \sqrt{1 - \lambda^2} (1 - \lambda^2 p_1^2)^{-1/2} p_1 \\ \sqrt{1 - \lambda^2} (1 - \lambda^2 p_1^2)^{-1/2} p_1 & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} 1 & \sqrt{1 - \lambda^2} p_1 (1 - \lambda^2 p_1^2)^{1/2} \\ \sqrt{1 - \lambda^2} p_1 (1 - \lambda^2 p_1^2)^{1/2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & (1 - \lambda^2 p_1^2)^{-1/2} p_0 \\ (1 - \lambda^2 p_1^2)^{-1/2} p_0 & 1 \end{pmatrix} x_0 \begin{pmatrix} 0 & p_0 (1 - \lambda^2 p_1^2)^{-1/2} \\ p_0 (1 - \lambda^2 p_1^2)^{-1/2} & 1 \end{pmatrix}.
\]

In the expression for \( x \), the (matrix) coefficients on \( x(\lambda) \) are invertible for all \( \lambda \in (0, 1) \). By grouping together the \( 2 \times 2 \) upper left corner blocks of the coefficients in the expression for \( x \), that is,
\[
x = \begin{pmatrix} a(\lambda) & 0 \\ 0 & 0 \end{pmatrix} x_{\alpha_1} \begin{pmatrix} a(\lambda) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b(\lambda) & 0 \\ 0 & 1 \end{pmatrix} x(\lambda) \begin{pmatrix} b(\lambda) & 0 \\ 0 & 1 \end{pmatrix},
\]
we see that Technique B applies once again (though cosmetically modified since 1 appears in the bottom right corner rather than in the top left) to yield \( x \) as a proper \( C^* \)-convex combination
\[
x = t(\lambda)^*x^{''}t(\lambda) + s(\lambda)^*x(\lambda)s(\lambda).
\]
(This is a proper combination because \( b(\lambda) \) is invertible for all \( \lambda \in (0, 1) \).) Because \( x \) is \( C^* \)-extreme in \( K \), \( x(\lambda) \sim x \) for all \( \lambda \in (0, 1) \). Moreover, because \( x(\lambda) \) depends continuously on \( \lambda \) and because the unitary orbit of \( x \) is closed, \( x(1) \sim x \). This proves that \( x \) is reducible, for \( x(1) \) is given by
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
x_{a_1}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix},
\]
or equivalently, \( x \sim x(1) = q_1x_{a_1}q_1 + q_1^+x_0q_1^+ \), where \( q_1 \) is a projection.

Recall that Technique C has \( x_0 \) as a \( C^* \)-convex combination of \( x_{a_2}, \ldots, x_{a_m} \) and so
\[
x(1) = q_1x_{a_1}q_1 + \sum_{i \geq 2} s_i^*x_{a_i}s_i,
\]
where \( q_1 + \sum_{i \geq 2} s_i^*s_i = 1 \) and \( q_1 \perp s_i \) for all \( i \geq 2 \). Observe that \( \sum_{i \geq 2} s_i^*s_i = 1 \mid_{\text{Im} q_1^+} \) and that the restriction of \( x(1) \) to \( \text{Im} q_1^+ \) is \( C^* \)-extreme in \( q_1^+Kq_1^+ \) (by the second of our two elementary facts stated prior to the theorem); hence, we may repeat our earlier arguments to conclude that one of the \( s_i \) is of norm 1, say \( s_2 \), and to conclude further that there exists a projection \( q_2 \) such that
\[
x \sim q_1x_{a_1}q_1 + q_2x_{a_2}q_2 + \text{(other terms)}.
\]
By exhausting this process after a finite number of steps, we see that \( x \sim \sum_i q_i x_{a_i}q_i \) for some projections \( q_i \) satisfying \( \sum_i q_i = 1 \). \( \square \)

4.2. Corollary. If \( \mathcal{P} \subset M_n \) is a compact set of normals, then \( K = C^* \text{-conv} \mathcal{P} \) has no nonnormal \( C^* \)-extreme points. Moreover, if \( X \subset \mathbb{C} \) is the convex hull of the eigenvalues of the elements of \( \mathcal{P} \), then \( x \in K \) is \( C^* \)-extreme in \( K \) if and only if \( x \) is normal and the eigenvalues of \( x \) are extreme points of \( X \).

Proof. It is easy to see (e.g., [4]) that \( K = C^* \text{-conv} \{ \zeta 1 : \zeta \text{ is an extreme point of } X \} \). If \( x \in K \) is \( C^* \)-extreme in \( K \), then Theorem 4.1 shows that \( x \) is a scalar matrix \( \zeta 1 \), or is reducible and of the form \( x = \sum q_i (\zeta_i 1)q_i \) for some projections \( q_i \) satisfying \( \sum q_i = 1 \). In either case, \( x \) is normal and the eigenvalues of \( x \) are extreme points of \( X \). Conversely, every normal \( x \in M_n \) with eigenvalues in the set of extreme points of \( X \) is \( C^* \)-extreme in \( K \) [4, Theorem 2]. \( \square \)

In [4, Theorem 3] it was shown that the nilpotent Jordan matrix of order \( n \) is an extreme point of \( K = C^* \text{-conv} \{ \zeta 1_n : \zeta^{n+1} = 1 \} \subset M_n \). Therefore, we now have a new set of examples, valid in each matrix algebra \( M_n \), of \( C^* \)-convex sets having linear extreme points that are not \( C^* \)-extreme. (Previously, the only known example was in \( M_2 \) [5].)

The methods of this paper also allow us to recover all of the \( C^* \)-extreme points found by Hopenwasser, Moore, and Paulsen of the \( C^* \)-convex set in \( M_2 \) of matrices with numerical radius not exceeding 1. As is demonstrated by Arveson [1, p. 302], this set is generated by \( \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \). In fact, we have the following general result.
4.3. **Theorem.** If $x \in M_2$, then the $C^*$-extreme points of $C^*$-$\text{conv}\{x\}$ are all of the matrices unitarily equivalent to $x$ and all of the scalar matrices obtained from the extreme points of the numerical range of $x$.

**Proof.** If $x \in M_2$ is normal, then Corollary 4.2 yields the result. If $x \in M_2$ is nonnormal, then $x$ is irreducible and so elements of the unitary orbit of $x$ are $C^*$-extreme in $C^*$-$\text{conv}\{x\}$ (by Theorem 3.1), as are all scalar matrices $\zeta I$ for which $\zeta$ is an extreme point of $W^1(x)$ [4, Theorem 1]. Conversely, suppose that $z$ is $C^*$-extreme in $C^*$-$\text{conv}\{x\}$. If $z$ is irreducible, then $z \sim x$; this follows from Theorem 4.1. If $z$ is reducible, then it must be unitarily equivalent to a diagonal matrix, so let us say that

$$z = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$ 

If $\alpha \neq \beta$, then there exist linearly independent unit vectors $\xi, \eta \in \mathbb{C}^2$ such that $(z\xi, \xi) = \alpha$ and $(z\eta, \eta) = \beta$. The matrices

$$t_1 = \sqrt{2}^{-1}[\xi, -\eta] \in M_2, \quad t_2 = \sqrt{2}^{-1}[\xi, \eta] \in M_2$$

are invertible and satisfy $t_1^*t_1 + t_2^*t_2 = I$ and $t_1^*xt_1 + t_2^*xt_2 = z$. But because $z$ is $C^*$-extreme in $C^*$-$\text{conv}\{x\}$, the (reducible) matrix $z$ must be unitarily equivalent to the (irreducible) matrix $x$, thereby yielding a contradiction. Thus, $z$ must be of the form $(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix})$. Plainly, in order for the scalar matrix $\alpha I$ to be extreme, let alone $C^*$-extreme, in $W^2(x)$, it is necessary that $\alpha$ be extreme in $W(x)$. \qed

5. **Matricial ranges of a hyponormal operator**

In their announcement [8], Pearcy and Salinas define the $n \times n$ reduc-\-ing matricial spectrum of $x \in B(H)$ to be the (possibly void) set $R^n(x)$ of $n \times n$ matrices of the form $\rho(x)$, where $\rho : C^*(x) \rightarrow M_n$ is a unital $*$-homomorphism and $C^*(x)$ is the unital $C^*$-algebra generated by $x$. By [8, Theorem 2], $\lambda \in R^1(x)$ if and only if there exists a sequence of unit vectors $\xi_j \in H$ satisfying $\lim_j \|(x - \lambda I)\xi_j\| = \lim_j \|(x - \lambda I)^*\xi_j\| = 0$. Therefore, if $x$ is hyponormal, meaning that $x^*x - xx^* \geq 0$, then $R^1(x)$ is precisely the approximate point spectrum $\sigma_a(x)$ of $x$, and it is $\sigma(x)$ if $x$ is in fact normal. Thus, the well-known result $W^1(x) = \text{conv} \sigma(x)$ for hyponormal $x$ leads to $W^1(x) = \text{conv} R^1(x)$ (by making use of the fact that the extreme points of $\text{conv} \sigma(x)$ must be approximate eigenvalues). An interesting theorem of Stampfli goes even further: if $\lambda = (x\xi, \xi)$ is an extreme point of $W^1(x)$ for some unit vector $\xi \in H$, then $x\xi = \lambda\xi$ and $x^*\xi = \lambda^*\xi$ [10].

The purpose of this section is to show that these numerical range properties of hyponormal operators are present at the level of matricial ranges.

5.1. **Theorem.** The following statements hold for every hyponormal operator $x$ acting on a Hilbert space $H$.

1. $W^n(x) = C^*$-$\text{conv} R^n(x)$.
2. The $C^*$-extreme points of $W^n(x)$ are elements of $R^n(x)$.
3. If $\Lambda = v^*xv$ is a $C^*$-extreme point of $W^n(x)$ for some isometry $v : C^n \rightarrow H$, then $vv^*$ is an invariant projection of $x$ and $x^*$. 

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Proof. (1) It is slightly more convenient to translate \( x \) by a scalar \( \zeta \) so that the new hyponormal operator \( \tilde{x} = x - \zeta I \) has \( 0 \in \sigma_f(\tilde{x}) \). Because the effect of translation on the matricial spectrum and the matricial range is just translation by the scalar matrix \( \zeta 1_n \), we assume without loss of generality that already \( 0 \in \sigma_f(x) \). Because \( x \) is hyponormal, the dilation theorem [11] of Sz.-Nagy and Foiaş asserts that there exists a normal operator \( z \) on a Hilbert space \( H' \) and a contraction \( a : H \rightarrow H' \) such that \( x = a^*za \) and \( \sigma(z) \subset \sigma_f(x) \). The spectral inclusion as stated is in fact \( R^1(z) \subset R^1(x) \), due to the normality of \( z \) and the hyponormality of \( x \). If \( R^1(z) \) does not contain \( 0 \), we can pass to the normal operator \( z \oplus 0 \) on \( H' \oplus \mathbb{C} \), having \( 0 \) in its spectrum, so that \( x \) is now of the form \( a_0^*(z \oplus 0)a_0^* \), where \( a_0 \) is the contraction obtained by the composition \( a_0 = pqa \) with \( q \) denoting the canonical injection \( H' \rightarrow H' \oplus \mathbb{C} \) and \( p \) denoting the canonical projection \( H' \oplus \mathbb{C} \rightarrow H' \). Hence, again without loss of generality, we will assume that already \( 0 \in R^1(z) \).

We first show that \( W^n(x) \subset W^n(z) \). Because \( 0 \in (R^1(z) \oplus 1_n) \), it follows that \( \lambda^*W^n(z)\lambda \subset W^n(z) \) for every contraction \( \lambda \in M_n \), by Remark 7 of [6]. Suppose now that \( \phi \) is a unital completely positive map \( C^*(x) \rightarrow M_\mathbb{C} \); we will show that \( \phi(x) \in W^n(z) \). The argument runs similar to that of Lemma 4.2 in [9]. Consider the completely positive map \( \psi : C^*(z) \rightarrow M_\mathbb{C} \) defined by \( \psi(f) = \phi(a^*fa) \), with \( \lambda = \psi(1) \). For the positive contraction \( \lambda \) there exists \( b \geq 0 \) in \( M_\mathbb{C} \) such that if \( p \) is the projection of \( \mathbb{C}^n \) onto the range of \( \lambda \), then \( b\lambda b = p \) (we have \( b = \lambda^{-1/2} \) on the range of \( p \)). Choose any state \( \theta \) on \( C^*(z) \) and consider the unital completely positive map \( \omega : C^*(z) \rightarrow M_\mathbb{C} \) defined by \( \omega(f) = b\psi(f)b + \theta(f)(1 - p) \). From \( \lambda^{1/2}\omega(z)\lambda^{1/2} = \phi(x) \) it follows that

\[
\phi(x) \in \lambda^{1/2}W^n(z)\lambda^{1/2} \subset W^n(z),
\]

as desired.

Because the sets \( R^n(x) \) and \( R^n(z) \) consist solely of normal matrices, their \( C^* \)-convex hulls are determined by their eigenvalue sets; that is, \( R^1(x) \) and \( R^1(z) \) determine the \( C^* \)-convex hulls of \( R^n(x) \) and \( R^n(z) \), respectively. Hence,

\[
W^n(z) = C^*-\text{conv } R^1(z) \subset C^*-\text{conv } R^1(x) \subset W^n(x) \subset W^n(z).
\]

(2) By (1), \( W^n(x) \) is the \( C^* \)-convex hull of a compact set of normals; hence, if \( \Lambda \in W^n(x) \) is a \( C^* \)-extreme point of \( W^n(x) \), then Corollary 4.2 has that \( \Lambda \) must be normal and its eigenvalues must be extreme in the convex hull of the spectra of \( R^n(x) \) (viz., the eigenvalues of \( \Lambda \) are extreme points of \( \text{conv } R^1(x) = W^1(x) \)). Each eigenvalue of \( \Lambda \) determines a unital \( * \)-homomorphism \( C^*(x) \rightarrow \mathbb{C} \) and by passing to a direct sum of these homomorphisms, we obtain a unital \( * \)-homomorphism \( \rho : C^*(x) \rightarrow M_n \) such that \( \rho(x) \) is unitarily equivalent to \( \Lambda \).

(3) By (2), the \( C^* \)-extreme point \( \Lambda = v^*xv \) is normal and \( \sigma(\Lambda) \) is a subset of the extreme points of \( W^1(x) \). Thus, whenever \( \xi \) is a unit eigenvector of \( \Lambda \) with corresponding eigenvalue \( \lambda \), then \( \lambda = (xv^*\xi, v^*\xi) \) is an extreme point of \( W^1(x) \). By Stampfli's theorem, \( x(v^*\xi) = \lambda(v^*\xi) \) and \( x^*(v^*\xi) = \lambda^*(v^*\xi) \). Furthermore, as \( \Lambda \) is completely determined by its spectral structure, we conclude that \( xv = v\Lambda \) and \( x^*v = v\Lambda^* \), whence \( vv^* \) commutes with \( x \). □
REFERENCES