THE COMMUTANT OF A CERTAIN COMPRESSION

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Abstract. Let $G$ be any bounded region in the complex plane and $K \subset G$ be a simple compact arc of class $C^1$. Let $A^2(G\setminus K)$ (resp. $A^2(G)$) be the Bergman space on $G\setminus K$ (resp. $G$). Let $S$ be the operator multiplication by $z$ on $A^2(G\setminus K)$ and $C = P_\mathcal{V}S|_{\mathcal{V}}$ be the compression of $S$ to the semi-invariant subspace $\mathcal{V} = A^2(G\setminus K) \otimes A^2(G)$. We show that the commutant of $C^*$ is the set of all operators of the form $A^{-1}M_hA$, where $h$ is a multiplier on a certain Sobolev space of functions on $K$ and $(Af)(w) = \int_G f(z)(\overline{z} - w)^{-1} \ dA(z)$ ($w \in K$). We also use multiplier theory in fractional order Sobolev spaces to obtain further information about $C$.

1. Introduction

Let $U$ be a bounded region in the complex plane and define $A^2(U)$, the Bergman space, to be the Hilbert space of analytic functions on $U$ with $\int_U |f(z)|^2 \ dA(z) < \infty$. (Here $dA$ is two-dimensional Lebesgue measure.) Define the subnormal multiplication operator $S^U$ on $A^2(U)$ by

$$(S^Uf)(z) = zf(z).$$

It is known [C2] that the commutant of $S^U$ (i.e., the set of operators on $A^2(U)$ that commute with $S^U$) is the set of operators of the form $S^U_\varphi$, where

$$(S^U_\varphi f)(z) = \varphi(z)f(z)$$

and $\varphi \in H^\infty(U)$, the algebra of bounded analytic functions on $U$. We mention that multiplication (and Toeplitz) operators on the Bergman space have been studied in [Ax, ACM] where it is shown that the spectrum $\sigma(S^U_\varphi)$ is the closure of $\varphi(U)$ and the essential spectrum $\sigma_e(S^U_\varphi)$ is the cluster set of $\varphi$ near the non-removable singularities of $\partial U$. (Here a point $\lambda \in \partial U$ is removable if there exists a neighborhood $W$ of $\lambda$ such that every $f \in A^2(U)$ has an analytic extension to $W \cup U$.)

Let $G$ be a bounded region in the complex plane and $K \subset G$ be a simple compact arc of class $C^1$. (Here $C^1$ means that $K$ has a continuously differentiable parameterization $\alpha: [a, b] \to K$ with $\alpha'(t) \neq 0$ on $[a, b]$.) The
Bergman space $A^2(G\setminus K)$ can be decomposed as

$$A^2(G\setminus K) = A^2(G) \oplus \mathcal{N},$$

which allows one to decompose $S^{G\setminus K}$ in matricial form as

$$S^{G\setminus K} = \begin{pmatrix} S^G & B \\ 0 & C \end{pmatrix}.$$ 

The object of study here is the semi-invariant subspace $\mathcal{N}$ and the compression $C = P_\mathcal{N}S^{G\setminus K}|_{\mathcal{N}}$ of $S^{G\setminus K}$ to $\mathcal{N}$. In [R1] it was shown that if $G$ is a Jordan domain, then every $f \in \mathcal{N}$ has an analytic continuation across the analytic arcs of $\partial G$, the operators $B$ and $C^*C - CC^*$ are compact, and $\sigma_e(C) = K$. (Here $\sigma(C)$ is the spectrum and $\sigma_e(C)$ is the essential spectrum of $C$.) We mention that a rough analog for the compression $C$ was studied for the Hardy space $H^p(G\setminus K)$ where similar properties were observed, see [C3, §§4 and 5].

In [R2] the lattice of invariant subspaces for $C$ was discussed by making the observation that every invariant subspace of $C$ is of the form $\mathcal{M} \ominus A^2(G)$, where $\mathcal{M}$ is $S^{G\setminus K}$-invariant and contains $A^2(G)$ and then by looking at such $\mathcal{M}$. In the same paper, it was shown how to represent $C^*$ as a multiplication operator on a certain Sobolev space of functions on $K$. Define $\mathcal{H}(K)$ as the space of functions in $L^2(K, |dz|)$ ($|dz|$ is arc length measure on $K$) with the following norm finite:

$$\|h\|_{\mathcal{H}(K)}^2 = \int_K |h(z)|^2 |dz| + \int_K \int_K \frac{|h(z) - h(w)|^2}{|z - w|^2} |dz| |dw|.$$ 

For $f \in L^2(G)$, define the conjugate-Cauchy transform $\hat{f}$ of $f$ by

$$\hat{f}(w) = \int_G \frac{f(z)}{z - w} A(z).$$

In [R2] it was shown that the map $(Af)(w) = \hat{f}(w)$ is an invertible operator from $\mathcal{N}$ onto $\mathcal{H}(K)$ with $(AC^*A^{-1}h)(w) = \overline{h}(w)$ for all $h \in \mathcal{H}(K)$. As a matter of fact, if for $\phi \in H^\infty(G)$ we define $C_\phi = P_\mathcal{N}S^{G\setminus K}|_{\mathcal{N}}$, then we can prove in a very similar way that $(AC_\psi^*A^{-1}h)(w) = \overline{\psi}(w)h(w)$ for all $h \in \mathcal{H}(K)$.

The purpose of this paper is to employ this representation of $C^*$ in order to compute the commutant of $C$. Since $\mathcal{N}$ is semi-invariant, a theorem of Sarason [S] gives us $C_\phi C_\psi = C_\psi C_\phi$ for all $\phi, \psi \in H^\infty(G)$. So knowing the commutant of $S^{G\setminus K}$, one might be led to believe that the commutant of $\mathcal{N}$ is the set of operators of the form $C_\phi$, where $\phi \in H^\infty(G)$. This, however, is not the case and we state our main theorem, which characterizes the commutant of $C$.

**Theorem 1.1.** If $Y$ is a continuous operator on $\mathcal{N}$ with $YC = CY$, then $Y^* = A^{-1}M_hA$, where $M_hk = hk$ and $h$ is a multiplier on the Sobolev space $\mathcal{H}(K)$.

The fact that $C^*$ can be represented as a multiplication operator and that $C^*C - CC^*$ is compact leads one to question whether or not $C$ is in fact a normal operator, or even similar to a normal operator. Using multiplier theory, as in work by Maz'ya and Shaposhnikova, we answer these two questions in the negative by the following theorem:

**Theorem 1.2.** The compression $C$ is not similar to a normal operator.
2. Preliminaries

For a bounded region $U$ in the complex plane, let $C^\infty_0(U)$ denote the set of infinitely differentiable functions with compact support in $U$. Define the Sobolev space $W^{2,0}_1(U)$ as the completion of $C^\infty_0(U)$ in the norm

$$
\|u\|_{W^{2,0}_1} = \left( \int_U |\nabla u|^2 \, dA(z) \right)^{1/2}.
$$

Let $B^2(U) = L^2(U) \oplus A^2(U)$ and note that by a result of [AFV] (also see [R2]), $B^2(U)$ is the $L^2(U)$-closure of $D^U C^\infty_0(U)$, where

$$
D^U = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right),
$$

and that $D^U$ extends to be a bounded invertible operator from $W^{2,0}_1(U)$ onto $B^2(U)$. It turns out [R2] that $T^U = (D^U)^{-1}$ can be given by $(T^U g)(w) = -\pi^{-1} \tilde{g}(w)$.

Since $zf, g = \langle f, zg \rangle$ for all $f, g \in L^2(U)$, we can define $R^U$ on $B^2(U)$ by

$$(R^U g)(w) = \overline{w} g(w).$$

We can also define $M^U$ on $W^{2,0}_1(U)$ by

$$(M^U h)(w) = \overline{w} h(w)$$

and see by [AFV] (see also [R2]) that $D^U M^U = R^U D^U$.

For our bounded region $G$ and $K \subset G$, a simple compact arc of class $C^1$, the differential operator $D^G$ will induce the obvious isomorphism

$$
\tilde{D}^G : W^{2,0}_1(G) / W^{2,0}_1(G \setminus K) \rightarrow B^2(G) / B^2(G \setminus K)
$$

by $\tilde{D}^G[h] = [D^G h]$. (Here and throughout the rest of this paper, we let $[j]$ denote the coset of the element $j$.) $R^G$ will induce a continuous operator $\tilde{R}^G$ on $B^2(G) / B^2(G \setminus K)$ by $\tilde{R}^G[g] = [R^G g]$ and $M^G$ will induce $\tilde{M}^G$ on $W^{2,0}_1(G) / W^{2,0}_1(G \setminus K)$ in a similar fashion to get

$$
\tilde{D}^G \tilde{M}^G = \tilde{R}^G \tilde{D}^G.
$$

Next, we define $X : W^{2,0}_1(G) \rightarrow \mathcal{H}(K)$ by $X h = \text{tr}(h)$, where $\text{tr}(h)$ is the trace of $h$ on $K$ as defined in [Liz] (also see [MS, Chapter 5]). (Roughly speaking, the trace is the 'restriction' of $h$ to $K$. For example, if $K = [a, b]$, an interval in the real line, we have $\text{tr}(h)(x) = \lim_{y \to 0} h(x, y)$ a.e. $[dx]$.) $X$ is a continuous surjective operator with ker$(X) = W^{2,0}_1(G \setminus K)$ (see [Liz, R2]), which will induce the continuous invertible operator

$$
\tilde{X} : W^{2,0}_1(G) / W^{2,0}_1(G \setminus K) \rightarrow \mathcal{H}(K)
$$

by $\tilde{X}[h] = \text{tr}(h)$.

Note that $\mathcal{N} = B^2(G) \oplus B^2(G \setminus K)$, so letting $i : \mathcal{N} \rightarrow B^2(G) / B^2(G \setminus K)$ be the natural map $i(f) = [f]$ we form $A : \mathcal{N} \rightarrow \mathcal{H}(K)$ by

$$
A = -\pi \tilde{X}(\tilde{D}^G)^{-1} i
$$

and see [R2] that $(Af)(w) = \tilde{f}(w)$ and $(AC^* A^{-1} h)(w) = \overline{w} h(w)$ for all $h \in \mathcal{H}(K)$.
3. Multipliers

For $\mathcal{B} = W^{1,0}_1(G)$ or $\mathcal{B} = \mathcal{H}(K)$ we define the space of multipliers on $\mathcal{B}$. We refer the reader to [MS] for a thorough reference on the subject of multipliers on Sobolev spaces.

A function $h \in \mathcal{B}$ is a multiplier if $gh \in \mathcal{B}$ for all $g \in \mathcal{B}$. An application of the closed graph theorem gives us that if $h$ is a multiplier, then the operator $M_h$ on $\mathcal{B}$ defined by $M_h g = hg$ is continuous. We denote the space of multipliers by $M\mathcal{B}$ and endow $M\mathcal{B}$ with the multiplier norm

$$||h||_{M\mathcal{B}} = \sup_{||g||_{\mathcal{B}} \leq 1} |gh|.$$ 

If $H \in MW^{2,0}_1(G)$ then $\text{tr}(H) \in M\mathcal{H}(K)$, and if $h \in M\mathcal{H}(K)$ then there exists an $H \in MW^{2,0}_1(G)$ with $\text{tr}(H) = h$. We refer the reader to [MS, Chapter 5; Str] for a discussion of these results.

This next lemma gives us an estimate for the norm of a multiplier in $\mathcal{H}(K)$, which will be used later. A proof can also be found in [MS, Chapter 2].

**Lemma 3.1.** If $h \in M\mathcal{H}(K)$, then

$$||h||_{M\mathcal{H}(K)} \sim \sup_{F \text{ compact}} \left( \frac{1}{\text{Cap}(F)} \right)^{1/2} \left( \iint_{G \setminus K} \frac{|h(z) - h(w)|^2}{|z - w|^2} |dz| |dw| \right)^{1/2} + ||h||_\infty.$$

Here $|| \cdot || \sim B$ means that there is a positive constant $c$ with $c^{-1}B \leq || \cdot || \leq cB$, and $\text{Cap}(F)$ denotes the logarithmic capacity of $F$.

4. Proof of Theorem 1.1

We now proceed to the proof of Theorem 1.1. Let $K$ be parameterized by $\alpha(t)$, $a \leq t \leq b$. We first show that polynomials in the independent variable $z$ are dense in $\mathcal{H}(K)$. The proof is quite simple, but we include it for the sake of completeness.

**Lemma 4.1.** Let $\psi \in C^\infty_0(G)$ and define $g(t) = \psi(\alpha(t))$. Then given $\epsilon > 0$, there exists a polynomial $p(z)$ with

$$\sup_{a \leq t \leq b} |p(\alpha(t)) - g(t)| < \epsilon$$

and

$$\sup_{a \leq t \leq b} \left| \frac{d}{dt}(p(\alpha(t)) - g(t)) \right| < \epsilon.$$ 

**Proof.** Since $G \setminus K$ is connected, then given $\epsilon > 0$, we apply Lavrentiev's theorem [C2] to find a polynomial $q(z)$ with

$$\sup_{z \in K} \left| q(z) - \frac{g'(\alpha^{-1}(z))}{\alpha'(\alpha^{-1}(z))} \right| < \epsilon.$$

Let $p(z)$ be a polynomial with $p' = q$ and $p(\alpha(a)) = 0$. Then

$$\frac{d}{dt} p(\alpha(t)) = q(\alpha(t)) \alpha'(t)$$
and
\[
\frac{d}{dt} (p(\alpha(t)) - g(t)) = q(\alpha(t))\alpha'(t) - g'(t) \\
= |\alpha'(t)| \left| q(\alpha(t)) - \frac{g'(t)}{\alpha'(t)} \right| \leq M \epsilon,
\]
where \( M = \sup\{|\alpha'(t)|: a \leq t \leq b\} \). Thus
\[
\sup_{a \leq t \leq b} |p(\alpha(t)) - g(t)| = \sup_{a \leq t \leq b} \left| \int_a^t \left( \frac{d}{ds} p(\alpha(s)) - g'(s) \right) ds \right| \leq (b - a) M \epsilon. \quad \Box
\]

Lemma 4.2. Polynomials in \( z \) are dense in \( \mathcal{H}(K) \).

Proof. Since the trace operator \( X \) is continuous, \( C_0^\infty(G)|_K \) is dense in \( \mathcal{H}(K) \). So let \( \psi \in C_0^\infty(G) \) and \( \epsilon > 0 \) be given. By Lemma 4.1, there is a polynomial \( p(z) \) with
\[
\sup_{a \leq t \leq b} |p(\alpha(t)) - \psi(\alpha(t))| \leq \epsilon
\]
and
\[
\sup_{a \leq t \leq b} \left| \frac{d}{dt} (p(\alpha(t)) - \psi(\alpha(t))) \right| \leq \epsilon.
\]
Let \( h(z) = p(z) - \psi(z) \). Then
\[
\|h\|^2_{\mathcal{H}(K)} = \int_K |h(z)|^2 |dz| + \int_K \int_K \frac{|h(z) - h(w)|^2}{|z - w|^2} |dz||dw| \\
= \int_a^b |h(\alpha(t))|^2 |\alpha'(t)|^2 dt \\
+ \int_a^b \int_a^b \frac{|h(\alpha(t)) - h(\alpha(s))|^2}{|\alpha'(t)|^2} |\alpha'(s)|^2 |dtdsdt| \\
\leq C \left( \int_a^b |h(\alpha(t))|^2 dt + \int_a^b \int_a^b \frac{|h(\alpha(t)) - h(\alpha(s))|^2}{|s - t|^2} dsdt \right) \leq C \epsilon. \quad \Box
\]

We now prove Theorem 1.1.

Proof. Let \( Y \) be a continuous operator on \( \mathcal{N} \) with \( YC = CY \). Then \( Y^*C^* = C^*Y^* \), so \( B = AY^*A^{-1} \) commutes with multiplication by \( \overline{z} \) on \( \mathcal{H}(K) \).

Let \( h = B(1) \) and note that \( B(\overline{p}) = B(\overline{p} \cdot 1) = \overline{p}h \) for all polynomials \( p(z) \). If \( f \in \mathcal{H}(K) \), choose a sequence of polynomials \( p_n \) with \( \overline{p}_n \rightarrow f \) in \( \mathcal{H}(K) \). We assume (by passing to a subsequence if necessary) that \( \overline{p}_n \rightarrow f \) a.e. on \( K \). Now \( B(\overline{p}_n) \rightarrow B(f) \) in \( \mathcal{H}(K) \) and \( B(\overline{p}_n) = h\overline{p}_n \) converges to \( hf \) a.e. on \( K \), so \( B(f) = hf \) a.e., making \( h \) a multiplier on \( \mathcal{H}(K) \). \( \Box \)

From this theorem and some multiplier theory, we can now express the commutant of \( C \) in terms of multipliers on \( W_1^{2,0}(G) \).

Corollary 4.1. Let \( Y \) be a continuous operator on \( \mathcal{N} \) with \( YC = CY \). Then
\[
Y^* = P_{\mathcal{N}} D^G M_H (D^G)^{-1}|_{\mathcal{N}}
\]
for some \( H \in MW_1^{2,0}(G) \).

Proof. By Theorem 1.1, \( Y^* = A^{-1} M_h A \) for some \( h \in M \mathcal{H}(K) \). So
\[
Y^* = i^{-1} \overline{D^G} \overline{X}^{-1} M_h \overline{X} \overline{D^G}^{-1} i.
\]
Letting \( H \in MW_{1}^{2,0}(G) \) with \( \text{tr}(H) = h \) we get that for any \( f \in \mathcal{N} \)

\[
Y^* f = i^{-1} \tilde{D}^G \hat{X}^{-1}(h \hat{f}) = i^{-1} \tilde{D}^G [H \hat{f}]
= i^{-1} [D^G M_H (D^G)^{-1} f] = P_x(D^G)^{-1} f.
\]

We also point out that as a consequence of Theorem 1.1, not every operator in the commutant of \( C \) is of the form \( C \psi \) for some \( \psi \in H^\infty(G) \). Let \( h \) be an infinitely differentiable function on \( K \) with no co-analytic extension to \( G \). Then if \( B = A^{-1} M_H A \), \( B^* \) commutes with \( C \). If \( B^* = C \psi \) for some \( \psi \in H^\infty(G) \), then using the fact that \( AC_\psi A^{-1} = M_\overline{\psi} \) on \( \mathcal{H}(K) \), we have \( h = \overline{\psi} \) on \( K \), which is a contradiction.

5. \( C \) IS NOT SIMILAR TO A NORMAL OPERATOR

The operator \( C^* \) can be represented as a multiplication operator on \( \mathcal{H}(K) \), and \( C^* C - CC^* \) is compact, which brings up the question of the normality of \( C \). We will now employ Lemma 3.1 and the spectral theorem for normal operators to prove that \( C \) is not similar to a normal operator.

**Proof of Theorem 1.2.** We prove this for \( K = [a, b] \) an interval in the real line and note that the proof for the general \( K \) is done in a similar manner.

If \( C \), equivalently \( C^* \), were similar to a normal operator, then by the above comments, \( M_x \) (multiplication by \( x \)) on \( \mathcal{H}([a, b]) \) would be similar to a normal operator. Without loss of generality, we can assume that \( [a, b] = [-1, 1] \). If \( M_x \) were similar to a normal operator, then by the spectral theorem for normal operators, we would have that for all \( \mu \notin [-1, 1] \),

\[
\|(M_x - \mu)^{-1}\| \leq C(\text{dist}(\mu, [-1, 1]))^{-1}
\]

for some \( C > 0 \) independent of \( \mu \). Let \( \mu = i\lambda \), \( 0 < \lambda < 1 \). We will show that

\[
\sup_{0<\lambda<1} \text{dist}(\mu, [-1, 1])\|(M_x - i\lambda)^{-1}\| = +\infty,
\]

which will rule out the normality (or similar to a normal) of \( M_x \) and hence \( C \). By Lemma 3.1, we know that

\[
\|(M_x - i\lambda)^{-1}\| \geq C \sup_{E \text{ compact}} \left( \frac{1}{\text{Cap}(E)} \right)^{1/2} \times \left( \int_E \int_E \frac{|1/(x - i\lambda) - 1/(y - i\lambda)|^2}{|x - y|^2} \, dx \, dy \right)^{1/2} + \left\| \frac{1}{x - i\lambda} \right\|_\infty
\]

\[
\geq C \sup_{0<r<1} \left( \frac{1}{\text{Cap}([-r, r])} \right)^{1/2} \left( \int_{-r}^r \int_{-r}^r \frac{dx \, dy}{|x - i\lambda|^2 |y - i\lambda|^2} \right)^{1/2} + \frac{1}{\lambda}.
\]

An estimate on logarithmic capacity (see [MS, Chapter 2]), yields

\[
\text{Cap}([-r, r]) \sim \left( \log \left( \frac{2}{r} \right) \right)^{-1}
\]

for all \( 0 < r < 1 \). Using this will give us that equation (3) is bounded below by

\[
C' \sup_{0<r<1} \left( \log \left( \frac{2}{r} \right) \right)^{1/2} \int_{-r}^r \frac{dx}{|x - i\lambda|^2} + \frac{1}{\lambda}.
\]
Since \( \text{dist}(i\lambda, [-1, 1]) = \lambda \), we can prove equation (1) by showing that

\[
(5) \quad \sup_{0 < \lambda < 1} \sup_{0 < r < 1} \lambda \left( \log \left( \frac{2}{r} \right) \right)^{1/2} \int_{-r}^{r} \frac{dx}{|x - i\lambda|^2} = +\infty.
\]

Note that

\[
\int_{-r}^{r} \frac{dx}{|x - i\lambda|^2} \geq \frac{2r}{|r - i\lambda|^2} = \frac{2r}{r^2 + \lambda^2},
\]

so the left-hand side of equation (5) is bounded below by

\[
(6) \quad \sup_{0 < \lambda < 1} \sup_{0 < r < 1} \lambda \left( \log \left( \frac{2}{r} \right) \right)^{1/2} \frac{2r}{r^2 + \lambda^2}.
\]

Letting \( r = \lambda \) we get that equation (6) is bounded below by

\[
\sup_{0 < r < 1} \left( \log \left( \frac{2}{r} \right) \right)^{1/2} = +\infty.
\]

Hence \( M_x \), and thus \( C \), is not similar to a normal operator. \( \square \)

References


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