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THE COMMUTANT OF A CERTAIN COMPRESSION

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ABSTRACT. Let G be any bounded region in the complex plane and $K \subset G$ be a simple compact arc of class C^1. Let $A^2(G\backslash K)$ (resp. $A^2(G)$) be the Bergman space on $G\backslash K$ (resp. G). Let S be the operator multiplication by z on $A^2(G\backslash K)$ and $C = P_{\mathcal{N}} S|_{\mathcal{N}}$ be the compression of S to the semi-invariant subspace $\mathcal{N} = A^2(G\backslash K) \ominus A^2(G)$. We show that the commutant of C^* is the set of all operators of the form $A^{-1} M_h A$, where h is a multiplier on a certain Sobolev space of functions on K and $(Af)(w) = \int_G f(z)(\overline{z} - \overline{w})^{-1}\, dA(z)$ $(w \in K)$. We also use multiplier theory in fractional order Sobolev spaces to obtain further information about C.

1. INTRODUCTION

Let U be a bounded region in the complex plane and define $A^2(U)$, the *Bergman space*, to be the Hilbert space of analytic functions on U with $\int_U |f(z)|^2\, dA(z) < \infty$. (Here dA is two-dimensional Lebesgue measure.) Define the subnormal multiplication operator S^U on $A^2(U)$ by

$$(S^U f)(z) = z f(z).$$

It is know [C2] that the commutant of S^U (i.e., the set of operators on $A^2(U)$ that commute with S^U) is the set of operators of the form S^U_φ, where

$$(S^U_\varphi f)(z) = \varphi(z) f(z)$$

and $\varphi \in H^\infty(U)$, the algebra of bounded analytic functions on U. We mention that multiplication (and Toeplitz) operators on the Bergman space have been studied in [Ax, ACM] where it is shown that the spectrum $\sigma(S^U_\varphi)$ is the closure of $\varphi(U)$ and the essential spectrum $\sigma_e(S^U_\varphi)$ is the cluster set of φ near the non-removable singularities of ∂U. (Here a point $\lambda \in \partial U$ is *removable* if there exists a neighborhood W of λ such that every $f \in A^2(U)$ has an analytic extension to $W \cup U$.)

Let G be a bounded region in the complex plane and $K \subset G$ be a simple compact arc of class C^1. (Here C^1 means that K has a continuously differentiable parameterization $\alpha\colon [a, b] \to K$ with $\alpha'(t) \neq 0$ on $[a, b]$.) The

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Bergman space $A^2(G\backslash K)$ can be decomposed as

$$A^2(G\backslash K) = A^2(G) \oplus \mathscr{N},$$

which allows one to decompose $S^{G\backslash K}$ in matricial form as

$$S^{G\backslash K} = \begin{pmatrix} S^G & B \\ 0 & C \end{pmatrix}.$$

The object of study here is the semi-invariant subspace $\mathscr{N}$ and the compression $C = P_{\mathscr{N}} S^{G\backslash K}|_{\mathscr{N}}$ of $S^{G\backslash K}$ to $\mathscr{N}$. In [R1] it was shown that if G is a Jordan domain, then every $f \in \mathscr{N}$ has an analytic continuation across the analytic arcs of ∂G, the operators B and $C^*C - CC^*$ are compact, and $\sigma_e(C) = K$. (Here $\sigma(C)$ is the spectrum and $\sigma_e(C)$ is the essential spectrum of C.) We mention that a rough analog for the compression C was studied for the Hardy space $H^p(G\backslash K)$ where similar properties were observed, see [C3, §§4 and 5].

In [R2] the lattice of invariant subspaces for C was discussed by making the observation that every invariant subspace of C is of the form $\mathscr{M} \ominus A^2(G)$, where $\mathscr{M}$ is $S^{G\backslash K}$-invariant and contains $A^2(G)$ and then by looking at such $\mathscr{M}$. In the same paper, it was shown how to represent C^* as a multiplication operator on a certain Sobolev space of functions on K. Define $\mathscr{H}(K)$ as the space of functions in $L^2(K, |dz|)$ ($|dz|$ is arc length measure on K) with the following norm finite:

$$\|h\|^2_{\mathscr{H}(K)} = \int_K |h(z)|^2 |dz| + \int_K \int_K \frac{|h(z) - h(w)|^2}{|z - w|^2} |dz| \, |dw|.$$

For $f \in L^2(G)$, define the conjugate-Cauchy transform $\tilde{f}$ of f by

$$\tilde{f}(w) = \int_G \frac{f(z)}{\overline{z} - \overline{w}} \, dA(z).$$

In [R2] it was shown that the map $(Af)(w) = \tilde{f}(w)$ is an invertible operator from $\mathscr{N}$ onto $\mathscr{H}(K)$ with $(AC^*A^{-1}h)(w) = \overline{w}h(w)$ for all $h \in \mathscr{H}(K)$. As a matter of fact, if for $\varphi \in H^\infty(G)$ we define $C_\varphi = P_{\mathscr{N}} S_\varphi^{G\backslash K}|_{\mathscr{N}}$, then we can prove in a very similar way that $(AC_\varphi^* A^{-1}h)(w) = \overline{\varphi(w)}h(w)$ for all $h \in \mathscr{H}(K)$.

The purpose of this paper is to employ this representation of C^* in order to compute the commutant of C. Since $\mathscr{N}$ is semi-invariant, a theorem of Sarason [S] gives us $C_\varphi C_\psi = C_\psi C_\varphi$ for all $\varphi, \psi \in H^\infty(G)$. So knowing the commutant of $S^{G\backslash K}$, one might be led to believe that the commutant of C is the set of operators of the form C_φ, where $\varphi \in H^\infty(G)$. This, however, is not the case and we state our main theorem, which characterizes the commutant of C.

Theorem 1.1. *If Y is a continuous operator on $\mathscr{N}$ with $YC = CY$, then $Y^* = A^{-1}M_hA$, where $M_hk = hk$ and h is a multiplier on the Sobolev space $\mathscr{H}(K)$.*

The fact that C^* can be represented as a multiplication operator and that $C^*C - CC^*$ is compact leads one to question whether or not C is in fact a normal operator, or even similar to a normal operator. Using multiplier theory, as in work by Maz'ya and Shaposhnikova, we answer these two questions in the negative by the following theorem:

Theorem 1.2. *The compression C is not similar to a normal operator.*

2. Preliminaries

For a bounded region U in the complex plane, let $C_0^\infty(U)$ denote the set of infinitely differentiable functions with compact support in U. Define the Sobolev space $W_1^{2,0}(U)$ as the completion of $C_0^\infty(U)$ in the norm

$$\|u\|_{W_1^{2,0}} = \left(\int_U |\nabla u|^2 \, dA(z)\right)^{1/2}.$$

Let $B^2(U) = L^2(U) \ominus A^2(U)$ and note that by a result of [AFV] (also see [R2]), $B^2(U)$ is the $L^2(U)$-closure of $D^U C_0^\infty(U)$, where

$$D^U = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$

and that D^U extends to be a bounded invertible operator from $W_1^{2,0}(U)$ onto $B^2(U)$. It turns out [R2] that $T^U = (D^U)^{-1}$ can be given by $(T^U g)(w) = -\pi^{-1}\tilde{g}(w)$.

Since $\langle zf, g\rangle = \langle f, \overline{z}g\rangle$ for all $f, g \in L^2(U)$, we can define R^U on $B^2(U)$ by

$$(R^U g)(w) = \overline{w}g(w).$$

We can also define M^U on $W_1^{2,0}(U)$ by

$$(M^U h)(w) = \overline{w}h(w)$$

and see by [AFV] (see also [R2]) that $D^U M^U = R^U D^U$.

For our bounded region G and $K \subset G$, a simple compact arc of class C^1, the differential operator D^G will induce the obvious isomorphism

$$\widetilde{D^G}\colon W_1^{2,0}(G)/W_1^{2,0}(G\backslash K) \to B^2(G)/B^2(G\backslash K)$$

by $\widetilde{D^G}[h] = [D^G h]$. (Here and throughout the rest of this paper, we let $[j]$ denote the coset of the element j.) R^G will induce a continuous operator $\widetilde{R^G}$ on $B^2(G)/B^2(G\backslash K)$ by $\widetilde{R^G}[g] = [R^G g]$ and M^G will induce $\widetilde{M^G}$ on $W_1^{2,0}(G)/W_1^{2,0}(G\backslash K)$ in a similar fashion to get

$$\widetilde{D^G}\widetilde{M^G} = \widetilde{R^G}\widetilde{D^G}.$$

Next, we define $X\colon W_1^{2,0}(G) \to \mathscr{H}(K)$ by $Xh = \operatorname{tr}(h)$, where $\operatorname{tr}(h)$ is the *trace* of h on K as defined in [Liz] (also see [MS, Chapter 5]). (Roughly speaking, the trace is the 'restriction' of h to K. For example, if $K = [a, b]$, an interval in the real line, we have $\operatorname{tr}(h)(x) = \lim_{y\to 0} h(x, y)$ a.e. $[dx]$.) X is a continuous surjective operator with $\ker(X) = W_1^{2,0}(G\backslash K)$ (see [Liz, R2]), which will induce the continuous invertible operator

$$\widetilde{X}\colon W_1^{2,0}(G)/W_1^{2,0}(G\backslash K) \to \mathscr{H}(K)$$

by $\widetilde{X}[h] = \operatorname{tr}(h)$.

Note that $\mathscr{N} = B^2(G) \ominus B^2(G\backslash K)$, so letting $i\colon \mathscr{N} \to B^2(G)/B^2(G\backslash K)$ be the natural map $i(f) = [f]$ we form $A\colon \mathscr{N} \to \mathscr{H}(K)$ by

$$A = -\pi\widetilde{X}(\widetilde{D^G})^{-1} i$$

and see [R2] that $(Af)(w) = \tilde{f}(w)$ and $(AC^*A^{-1}h)(w) = \overline{w}h(w)$ for all $h \in \mathscr{H}(K)$.

3. Multipliers

For $\mathscr{B} = W_1^{2,0}(G)$ or $\mathscr{B} = \mathscr{H}(K)$ we define the space of multipliers on $\mathscr{B}$. We refer the reader to [MS] for a thorough reference on the subject of multipliers on Sobolev spaces.

A function $h \in \mathscr{B}$ is a *multiplier* if $gh \in \mathscr{B}$ for all $g \in \mathscr{B}$. An application of the closed graph theorem gives us that if h is a multiplier, then the operator M_h on $\mathscr{B}$ defined by $M_h g = hg$ is continuous. We denote the space of multipliers by $M\mathscr{B}$ and endow $M\mathscr{B}$ with the multiplier norm

$$\|h\|_{M\mathscr{B}} = \sup_{\|g\|_{\mathscr{B}} \le 1} \|gh\|_{\mathscr{B}}.$$

If $H \in MW_1^{2,0}(G)$ then $\operatorname{tr}(H) \in M\mathscr{H}(K)$, and if $h \in M\mathscr{H}(K)$ then there exists an $H \in MW_1^{2,0}(G)$ with $\operatorname{tr}(H) = h$. We refer the reader to [MS, Chapter 5; Str] for a discussion of these results.

This next lemma gives us an estimate for the norm of a multiplier in $\mathscr{H}(K)$, which will be used later. A proof can also be found in [MS, Chapter 2].

Lemma 3.1. *If $h \in M\mathscr{H}(K)$, then*

$$\|h\|_{M\mathscr{H}(K)} \sim \sup_{F \text{ compact}} \left(\frac{1}{\operatorname{Cap}(F)}\right)^{1/2} \left(\int_F \int_K \frac{|h(z)-h(w)|^2}{|z-w|^2} |dz|\,|dw|\right)^{1/2} + \|h\|_\infty .$$

Here $\|\cdot\| \sim B$ means that there is a positive constant c with $c^{-1}B \le \|\cdot\| \le cB$, and $\operatorname{Cap}(F)$ denotes the logarithmic capacity of F.

4. Proof of Theorem 1.1

We now proceed to the proof of Theorem 1.1. Let K be parameterized by $\alpha(t)$, $a \le t \le b$. We first show that polynomials in the independent variable z are dense in $\mathscr{H}(K)$. The proof is quite simple, but we include it for the sake of completeness.

Lemma 4.1. *Let $\psi \in C_0^\infty(G)$ and define $g(t) = \psi(\alpha(t))$. Then given $\varepsilon > 0$, there exists a polynomial $p(z)$ with*

$$\sup_{a \le t \le b} |p(\alpha(t)) - g(t)| < \varepsilon$$

and

$$\sup_{a \le t \le b} \left| \frac{d}{dt}(p(\alpha(t)) - g(t)) \right| < \varepsilon .$$

Proof. Since $G \backslash K$ is connected, then given $\varepsilon > 0$, we apply Lavrientiev's theorem [C2] to find a polynomial $q(z)$ with

$$\sup_{z \in K} \left| q(z) - \frac{g'(\alpha^{-1}(z))}{\alpha'(\alpha^{-1}(z))} \right| < \varepsilon .$$

Let $p(z)$ be a polynomial with $p' = q$ and $p(\alpha(a)) = 0$. Then

$$\frac{d}{dt} p(\alpha(t)) = q(\alpha(t))\alpha'(t)$$

and

$$\left|\frac{d}{dt}(p(\alpha(t)) - g(t))\right| = |q(\alpha(t))\alpha'(t) - g'(t)|$$
$$= |\alpha'(t)|\left|q(\alpha(t)) - \frac{g'(t)}{\alpha'(t)}\right| \le M\varepsilon,$$

where $M = \sup\{|\alpha'(t)|: a \le t \le b\}$. Thus

$$\sup_{a\le t\le b} |p(\alpha(t)) - g(t)| = \sup_{a\le t\le b}\left|\int_a^t \left(\frac{d}{ds}p(\alpha(s)) - g'(s)\right) ds\right| \le (b-a)M\varepsilon. \quad \square$$

Lemma 4.2. *Polynomials in z are dense in $\mathscr{H}(K)$.*

Proof. Since the trace operator X is continuous, $C_0^\infty(G)|_K$ is dense in $\mathscr{H}(K)$. So let $\psi \in C_0^\infty(G)$ and $\varepsilon > 0$ be given. By Lemma 4.1, there is a polynomial $p(z)$ with

$$\sup_{a\le t\le b} |p(\alpha(t)) - \psi(\alpha(t))| \le \varepsilon$$

and

$$\sup_{a\le t\le b}\left|\frac{d}{dt}(p(\alpha(t)) - \psi(\alpha(t)))\right| \le \varepsilon.$$

Let $h(z) = p(z) - \psi(z)$. Then

$$\begin{aligned}
\|h\|^2_{\mathscr{H}(K)} &= \int_K |h(z)|^2\,|dz| + \int_K\int_K \frac{|h(z)-h(w)|^2}{|z-w|^2}\,|dz|\,|dw| \\
&= \int_a^b |h(\alpha(t))|^2|\alpha'(t)|^2\,dt \\
&\quad + \int_a^b\int_a^b \frac{|h(\alpha(t)) - h(\alpha(s))|^2}{|\alpha(t)-\alpha(s)|^2}|\alpha'(t)||\alpha'(s)|\,ds\,dt \\
&\le C\left(\int_a^b |h(\alpha(t))|^2\,dt + \int_a^b\int_a^b \frac{|h(\alpha(t)) - h(\alpha(s))|^2}{|s-t|^2}\,ds\,dt\right) \le C'\varepsilon. \quad \square
\end{aligned}$$

We now prove Theorem 1.1.

Proof. Let Y be a continuous operator on $\mathscr{N}$ with $YC = CY$. Then $Y^*C^* = C^*Y^*$, so $B = AY^*A^{-1}$ commutes with multiplication by $\overline{z}$ on $\mathscr{H}(K)$.

Let $h = B(1)$ and note that $B(\overline{p}) = B(\overline{p}\cdot 1) = \overline{p}h$ for all polynomials $p(z)$. If $f \in \mathscr{H}(K)$, choose a sequence of polynomials p_n with $\overline{p}_n \to f$ in $\mathscr{H}(K)$. We assume (by passing to a subsequence if necessary) that $\overline{p}_n \to f$ a.e. on K. Now $B(\overline{p}_n) \to B(f)$ in $\mathscr{H}(K)$ and $B(\overline{p}_n) = h\overline{p}_n$ converges to hf a.e. on K, so $B(f) = hf$ a.e., making h a multiplier on $\mathscr{H}(K)$. $\square$

From this theorem and some multiplier theory, we can now express the commutant of C in terms of multipliers on $W_1^{2,0}(G)$.

Corollary 4.1. *Let Y be a continuous operator on $\mathscr{N}$ with $YC = CY$. Then*

$$Y^* = P_{\mathscr{N}} D^G M_H (D^G)^{-1}|_{\mathscr{N}}$$

for some $H \in MW_1^{2,0}(G)$.

Proof. By Theorem 1.1, $Y^* = A^{-1}M_hA$ for some $h \in M\mathscr{H}(K)$. So

$$Y^* = i^{-1}\widetilde{D}^G\widetilde{X}^{-1}M_h\widetilde{X}\widetilde{D}^{G^{-1}}i.$$

Letting $H \in MW_1^{2,0}(G)$ with $\operatorname{tr}(H) = h$ we get that for any $f \in \mathcal{N}$

$$\begin{aligned} Y^* f &= i^{-1}\widetilde{D}^G \widetilde{X}^{-1}(h\tilde{f}) = i^{-1}\widetilde{D}^G[H\tilde{f}] \\ &= i^{-1}[D^G M_H (D^G)^{-1} f] = P_{\mathcal{N}}(D^G)^{-1} f. \quad \square \end{aligned}$$

We also point out that as a consequence of Theorem 1.1, not every operator in the commutant of C is of the form C_ψ for some $\psi \in H^\infty(G)$. Let h be an infinitely differentiable function on K with no co-analytic extension to G. Then if $B = A^{-1}M_h A$, B^* commutes with C. If $B^* = C_\psi$ for some $\psi \in H^\infty(G)$, then using the fact that $AC_\psi^* A^{-1} = M_{\overline{\psi}}$ on $\mathcal{H}(K)$, we have $h = \overline{\psi}$ on K, which is a contradiction.

5. C IS NOT SIMILAR TO A NORMAL OPERATOR

The operator C^* can be represented as a multiplication operator on $\mathcal{H}(K)$, and $C^*C - CC^*$ is compact, which brings up the question of the normality of C. We will now employ Lemma 3.1 and the spectral theorem for normal operators to prove that C is not similar to a normal operator.

Proof of Theorem 1.2. We prove this for $K = [a, b]$ an interval in the real line and note that the proof for the general K is done in a similar manner.

If C, equivalently C^*, were similar to a normal operator, then by the above comments, M_x (multiplication by x) on $\mathcal{H}([a, b])$ would be similar to a normal operator. Without loss of generality, we can assume that $[a, b] = [-1, 1]$. If M_x were similar to a normal operator, then by the spectral theorem for normal operators, we would have that for all $\mu \notin [-1, 1]$,

$$\|(M_{x-\mu})^{-1}\| \le C(\operatorname{dist}(\mu, [-1, 1]))^{-1}$$

for some $C > 0$ independent of μ. Let $\mu = i\lambda$, $0 < \lambda < 1$. We will show that

$$\sup_{0<\lambda<1} \operatorname{dist}(\mu, [-1, 1])\|(M_{x-i\lambda})^{-1}\| = +\infty, \tag{1}$$

which will rule out the normality (or similar to a normal) of M_x and hence C. By Lemma 3.1, we know that

$$\begin{aligned} \|(M_{x-i\lambda})^{-1}\| &\ge C \sup_{E \text{ compact}} \left(\frac{1}{\operatorname{Cap}(E)}\right)^{1/2} \\ &\quad \times \left(\int_E \int_E \frac{|1/(x-i\lambda) - 1/(y-i\lambda)|^2}{|x-y|^2}\, dx\, dy\right)^{1/2} + \left\|\frac{1}{x-i\lambda}\right\|_\infty \qquad (2) \\ &\ge C \sup_{0<r<1} \left(\frac{1}{\operatorname{Cap}([-r, r])}\right)^{1/2} \left(\int_{-r}^{r}\int_{-r}^{r} \frac{dx\, dy}{|x-i\lambda|^2 |y-i\lambda|^2}\right)^{1/2} + \frac{1}{\lambda}. \qquad (3) \end{aligned}$$

An estimate on logarithmic capacity (see [MS, Chapter 2]), yields

$$\operatorname{Cap}([-r, r]) \sim \left(\log\left(\frac{2}{r}\right)\right)^{-1}$$

for all $0 < r < 1$. Using this will give us that equation (3) is bounded below by

$$C' \sup_{0<r<1} \left(\log\left(\frac{2}{r}\right)\right)^{1/2} \int_{-r}^{r} \frac{dx}{|x-i\lambda|^2} + \frac{1}{\lambda}. \tag{4}$$

Since $\operatorname{dist}(i\lambda, [-1, 1]) = \lambda$, we can prove equation (1) by showing that

$$\sup_{0<\lambda<1} \sup_{0<r<1} \lambda \left(\log\left(\frac{2}{r}\right)\right)^{1/2} \int_{-r}^{r} \frac{dx}{|x - i\lambda|^2} = +\infty. \tag{5}$$

Note that

$$\int_{-r}^{r} \frac{dx}{|x - i\lambda|^2} \geq \frac{2r}{|r - i\lambda|^2} = \frac{2r}{r^2 + \lambda^2},$$

so the left-hand side of equation (5) is bounded below by

$$\sup_{0<\lambda<1} \sup_{0<r<1} \lambda \left(\log\left(\frac{2}{r}\right)\right)^{1/2} \frac{2r}{r^2 + \lambda^2}. \tag{6}$$

Letting $r = \lambda$ we get that equation (6) is bounded below by

$$\sup_{0<r<1} \left(\log\left(\frac{2}{r}\right)\right)^{1/2} = +\infty.$$

Hence M_x, and thus C, is not similar to a normal operator. □

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