PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

AN APPLICATION OF SET THEORY TO THE TORSION PRODUCT OF ABELIAN GROUPS

PATRICK KEEF

(Communicated by Ron Solomon)

ABSTRACT. The following problem of Fuchs is considered: relate the abelian groups A and B assuming $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(B, G)$ for all reduced abelian groups G. A complete characterization is obtained in any set-theoretic universe in which $E(\kappa)$ is valid for a proper class of regular cardinals κ.

In the well-known list of 100 questions posed in [3], problem 50 asks us to relate the abelian groups A and B assuming $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(B, G)$ for all reduced abelian groups G. There is clearly no loss of generality in assuming A, B, and G are p-groups for some fixed prime p, so hereafter the term "group" will mean "abelian p-group." Hill showed in [4] that A and B share many properties but left unresolved whether they must necessarily be isomorphic. Cutler showed in [1] that this need not be the case. Specifically, groups A and C were found such that C is a direct sum of cyclics, $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(A \oplus C, G)$ for all reduced G, but $A \ncong A \oplus C$. Is this the only way to construct such examples? In other words, if A and B satisfy problem 50, do they necessarily differ by summands which are direct sums of cyclics?

Denote the Ulm function of the group Y by f_Y. We will say A and B are *T-equivalent* if

(a) $f_A(n) = f_B(n)$ for all $n < \omega$, and
(b) for some direct sums of cyclics X and Y, $A \oplus X \cong B \oplus Y$.

We show that if A and B are T-equivalent, then they satisfy problem 50 (Theorem 2). Conversely, if A and B satisfy problem 50 and there is a regular cardinal κ greater than the ranks of A and B such the $E(\kappa)$ is valid, we show that A and B must be T-equivalent (Theorem 4; we review these set-theoretic terms later). In particular, we have a solution for problem 50 in any set-theoretic universe in which $E(\kappa)$ is valid for a proper class of regular cardinals. This will be true, for example, in the constructible universe.

We first review the treatment of Ulm invariants contained in [5]. If Z is a valuated group and α is an ordinal, let $k_Z(\alpha)$ be the kernel of the map $Z(\alpha)/Z(\alpha+1) \xrightarrow{\times p} Z(\alpha+1)/Z(\alpha+2)$, so $f_Z(\alpha)$ is the dimension of $k_Z(\alpha)$ as a

Received by the editors November 11, 1991.
1991 *Mathematics Subject Classification*. Primary 20K10, 20K40.
Key words and phrases. Abelian p-group, torsion product.
The author was supported by NSF (RUI) grant DMS-9000129.

0002-9939/93 $1.00 + $.25 per page

$\mathbb{Z}_p$-vector space. If Z is a subgroup of Y and, for $z \in Z$, $v(z) = \mathrm{ht}_Y(z)$, then the αth relative Ulm invariant, $f_{Y,Z}(\alpha)$, is the dimension of the cokernel of the induced injection $k_Z(\alpha) \to k_Y(\alpha)$. It follows that $f_Y(\alpha) = f_{Y,Z}(\alpha) + f_Z(\alpha)$. We apply this in

Lemma 1. *Suppose A and B are T-equivalent. Then there are direct sums of cyclics X and Y such that*

(a) $f_{A\oplus X}(n) = f_A(n) = f_B(n) = f_{B\oplus Y}(n)$ *for all* $n < \omega$, *and*
(b) $A \oplus X \cong B \oplus Y$.

Proof. For some direct sums of cyclics X' and Y', $A \oplus X' \cong B \oplus Y'$; we identify these and call the result H. Let $G = A \cap B$. Note that for all $g \in G$, $\mathrm{ht}_A(g) = \mathrm{ht}_H(g) = \mathrm{ht}_B(g)$, so $\mathrm{id}_G \colon G \to G$ preserves heights computed in A and B, respectively. Since A/G embeds in Y', it is a direct sum of cyclics; similarly, B/G is a direct sum of cyclics. Let $f = f_A = f_B$. If G is given the valuation induced by the height function on H, then for every $n < \omega$,

$$f_{A,G}(n) + f_G(n) = f(n) = f_{B,G}(n) + f_G(n).$$

Let X be a direct sum of cyclics such that $f_X(n) + f_{A,G}(n) = f_{B,G}(n)$ whenever $f_{A,G}(n) < f_{B,G}(n)$, and $f_X(n) = 0$ otherwise. Similarly, let Y be a direct sum of cyclics such that $f_Y(n) + f_{B,G}(n) = f_{A,G}(n)$ whenever $f_{B,G}(n) < f_{A,G}(n)$, and $f_Y(n) = 0$ otherwise. It follows that for all $n < \omega$,

$$f(n) + f_X(n) = f(n) = f(n) + f_Y(n)$$

so that (a) is valid. Further,

$$f_{A\oplus X, G\oplus 0}(n) = f_{A,G}(n) + f_X(n) = f_{B,G}(n) + f_Y(n) = f_{B\oplus Y, G\oplus 0}(n).$$

So by [3, 83.4], $\mathrm{id}_{G\oplus 0}$ extends to an isomorphism $A \oplus X \cong B \oplus Y$, as required.

Theorem 2. *If A and B are T-equivalent, then for every reduced group G,* $\mathrm{Tor}(A, G) \cong \mathrm{Tor}(B, G)$.

Proof. Let X and Y be as in Lemma 1. We will be done if we can show $\mathrm{Tor}(A, G) \cong \mathrm{Tor}(A\oplus X, G)$ for all reduced G, since similar isomorphisms will be true for B and $A \oplus X \cong B \oplus Y$. It is readily checked that our hypothesis guarantees that for every $n < \omega$, $r(p^n A) = r(p^n(A \oplus X))$. By [3, 64.4] for any groups M and N and ordinal α there is an equation

$$(*) \qquad f_{\mathrm{Tor}(M,N)}(\alpha) = f_M(\alpha) f_N(\alpha) + f_M(\alpha) r(p^{\alpha+1} N) + r(p^{\alpha+1} M) f_N(\alpha).$$

If $\mathrm{Tor}(A, G)$ is a direct sum of cyclic groups then so is $\mathrm{Tor}(A \oplus X, G)$, and it follows from $(*)$ that $\mathrm{Tor}(A, G) \cong \mathrm{Tor}(A \oplus X, G)$. In particular, if either A or G is bounded, the isomorphism follows. Suppose both A and G are unbounded. If Z is isomorphic to a basic subgroup of $\mathrm{Tor}(A, G)$, it follows from $(*)$ that $Z \cong Z \oplus Z$. By [6, 1.7] $\mathrm{Tor}(A, G) \cong \mathrm{Tor}(A, G) \oplus Z$. Since $\mathrm{Tor}(X, G)$ is a direct sum of cyclics and by $(*)$, $f_{\mathrm{Tor}(X,G)}(n) \le f_Z(n)$ for all $n < \omega$, it follows that $Z \cong Z \oplus \mathrm{Tor}(X, G)$. Therefore,

$$\begin{aligned}\mathrm{Tor}(A, G) &\cong \mathrm{Tor}(A, G) \oplus Z \cong \mathrm{Tor}(A, G) \oplus Z \oplus \mathrm{Tor}(X, G)\\ &\cong \mathrm{Tor}(A, G) \oplus \mathrm{Tor}(X, G) \cong \mathrm{Tor}(A \oplus X, G),\end{aligned}$$

as required.

We now review some notions from set theory. Suppose α is a limit ordinal. A subset $C \subseteq \alpha$ is a *cub* if it is closed (in the order topology) and unbounded. A subset $S \subseteq \alpha$ is *stationary* if for every cub $C \subseteq \alpha$, $S \cap C \neq \varnothing$. The *cofinality* of α is the minimal cardinality of its unbounded subsets. A cardinal κ is *regular* if it has cofinality κ. If κ is a regular cardinal, then $E(\kappa)$ is the following statement:

> There is a stationary subset $S \subseteq \kappa$ such that (1) every $\alpha \in S$ is a limit ordinal of countable cofinality, and (2) for every limit ordinal $\alpha < \kappa$, $\alpha \cap S$ is not a stationary subset of α.

If G is a group of cardinals, then a collection of subgroups $\{W_j\}_{j<\alpha}$ whose union is G is a *κ-filtration* if for every $j < \alpha$, $W_j = \bigcup_{i<j} W_{i+1}$ and $|W_j| < |G|$.

Theorem 3. *Suppose κ is a regular cardinal for which $E(\kappa)$ is valid and $S \subseteq \kappa$ is as in the definition of $E(\kappa)$. Then there is a group G of cardinality κ with a κ-filtration $\{W_j\}_{j<\kappa}$ consisting of pure subgroups, such that whenever $j < l < \kappa$ and $j \in S$, $W_l/W_j \cong \mathbb{Z}_{p^\infty} \oplus M$, where M is a direct sum of cyclics.*

Proof. For every $i < \kappa$ and $m < \omega$, let $C_m^i \cong \mathbb{Z}_{p^m}$ and $B^i = \bigoplus_{m<\omega} C_m^i$. For a moment, fix $j \in S$ and let $x_1, x_2, x_3, \dots$ be a strictly increasing sequence of isolated ordinals with limit j. Let $\bigoplus_{m<\omega} C_m^{x_m} \subseteq Z_j \subseteq \prod_{m<\omega} C_m^{x_m}$ satisfy $Z_j/\bigoplus_{m<\omega} C_m^{x_m} \cong \mathbb{Z}_{p^\infty}$. Let $G \subseteq \prod_{i<\kappa} B^i$ be generated by $\bigoplus_{i<\kappa} B^i$ and the Z_j's for $j \in S$. If $j < \kappa$, let $G_j = G \cap \prod_{i<j} B^i$ and $G^j = G \cap \prod_{j\leq i<\kappa} B^i$. Observe that if $j < \kappa$ and $\pi_j\colon \prod_{i<\kappa} B^i \to \prod_{i<j} B^i$ is the obvious projection, then whenever $l > j$ is in S, we have $\pi_j(Z_l) \subseteq \bigoplus_{i<j} B^i$. Therefore, $\pi_j(G) \subseteq G$, so that $G \cong G_j \oplus G^j$. In addition G_j will be generated by $\bigoplus_{i<j} B^i$ and the Z_i for $i \leq j$ with $i \in S$. In particular, $|G_j| = |j| \cdot \omega_0$. Finally, let $W_j = \bigcup_{i<j} G_i$. Clearly these form a κ-filtration of G by pure subgroups.

We claim that G_j is a direct sum of cyclics for all $j < \kappa$. We prove this by induction. Since $G_{j+1} \cong G_j \oplus B^j$, the interesting case is where j is a limit ordinal. Let $g\colon \mu \to j - S$ be a strictly increasing function whose image is a cub. For each $\beta < \mu$, $G_{g(\beta)}$ is a direct sum of cyclics which is a summand of G, hence, of $G_{g(\beta+1)}$. Note that if $\sigma < \mu$ is a limit, then $g(\sigma)$ is not in S, so that $G_{g(\sigma)} = W_{g(\sigma)}$. Therefore, $W_j \cong \bigoplus_{\beta<\mu} G_{g(\beta+1)}/G_{g(\beta)}$ is a direct sum of cyclics. Note that if $j \notin S$, then $G_j = W_j$ and the claim follows, and if $j \in S$, then G_j is generated by W_j, and Z_j, hence, $G_j/W_j \cong \mathbb{Z}_{p^\infty}$. There is a decomposition $W_j = K \oplus L$, where L is a countable subgroup containing $W_j \cap Z_j$. Clearly K is a direct sum of cyclics, and since $L + Z_j$ is countable, it is a direct sum of cyclics, also. Therefore $G_j = K \oplus (L + Z_j)$ is a direct sum of cyclics, as required.

Note that from the last paragraph, if $j \in S$ and $j < l < \kappa$, then

$$\mathbb{Z}_{p^\infty} \cong G_j/W_j \subseteq W_l/W_j \subseteq G_l/W_j,$$

and therefore $W_l/W_j \cong \mathbb{Z}_{p^\infty} \oplus M$, where $M \subseteq G_l/G_j$ is a direct sum of cyclics.

Let λ be the supremum of the values of κ for which $E(\kappa)$ is valid. So λ is either a cardinal or the symbol ∞ if $E(\kappa)$ is valid for a proper class of regular cardinals.

Observe that if $A[p] \cong \operatorname{Tor}(A, \mathbb{Z}_p) \cong \operatorname{Tor}(B, \mathbb{Z}_p) \cong B[p]$, then A and B have the same rank.

Theorem 4. *Suppose* $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(B, G)$ *for every reduced* G *and the rank of* A *and* B *is less than* λ. *Then* A *and* B *are* T*-equivalent.*

Proof. By [4, Theorem 1] A and B have the same Ulm invariants. Let κ be a regular cardinal greater than the ranks of A and B for which $E(\kappa)$ holds and let $S \subseteq \kappa$, G, and $\{W_j\}_{j<\kappa}$ be as in Theorem 3. Suppose g: $\operatorname{Tor}(A, G) \to \operatorname{Tor}(B, G)$ is an isomorphism. Note that $\{\operatorname{Tor}(A, W_j)\}_{j<\kappa}$, $\{\operatorname{Tor}(B, W_j)\}_{j<\kappa}$ are κ-filtrations of $\operatorname{Tor}(A, G)$, $\operatorname{Tor}(B, G)$, respectively. By a simple back-and-forth argument, there is a cub $C \subseteq \kappa$ such that for all $j \in C$, $g(\operatorname{Tor}(A, W_j)) = \operatorname{Tor}(B, W_j)$. Since S is stationary, there is a $j \in S \cap C$. Let $l > j$ with $l \in C$. Since W_j is a pure subgroup of W_l and Tor is exact on pure sequences [3, 63.2], it follows that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \operatorname{Tor}(A, W_j) & \to & \operatorname{Tor}(A, W_l) & \to & \operatorname{Tor}(A, W_l/W_j) & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\exists\cong & & \\
0 & \to & \operatorname{Tor}(B, W_j) & \to & \operatorname{Tor}(B, W_l) & \to & \operatorname{Tor}(B, W_l/W_j) & \to & 0
\end{array}$$

If $W_l/W_j \cong \mathbb{Z}_{p^\infty} \oplus M$, then $X = \operatorname{Tor}(A, M)$ and $Y = \operatorname{Tor}(B, M)$ are direct sums of cyclics and

$$A \oplus X \cong \operatorname{Tor}(A, W_l/W_j) \cong \operatorname{Tor}(B, W_l/W_j) \cong B \oplus Y,$$

which proves the result.

Corollary 5. *If* $\lambda = \infty$, *then* $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(B, G)$ *for all reduced* G *iff* A *and* B *are* T*-equivalent.*

We conclude with a discussion of the condition $\lambda = \infty$. As noted above, this is true in $V = L$. More generally, if there is a proper class of successor cardinals *in* L which are also cardinals *in* V, then $\lambda = \infty$ [2, VI, 3.6]. This holds, for example, when $0^\#$ does not exist [2, VI, 3.16]. The condition $\lambda < \infty$ is related to the existence of large cardinals of certain types. For example, if $\lambda < \infty$, then it is consistent that the measurable cardinals form a proper class (this follows from [2, VI, 3.17], since there are arbitrarily large singular cardinals). Conversely, if there is a $L_{\omega_1\omega}$-compact cardinal, then we can conclude that $\lambda < \infty$ [2, VI, 3.18]. Note that if $\lambda < \infty$, it is still possible that any two groups satisfying problem 50 are T-equivalent, but some other technique will have to be employed to prove this.

References

1. D. Cutler, *Abelian p-groups* A *and* B *such that* $\operatorname{Tor}(A, G) \cong \operatorname{Tor}(B, G)$, G *reduced*, Proc. Amer. Math. Soc. **91** (1984), 12–14.
2. P. Eklof and A. Mekler, *Almost free modules, set-theoretic methods*, North-Holland, Amsterdam, 1990.
3. L. Fuchs, *Infinite abelian groups*, Vols. 1 and 2, Academic Press, New York, 1970 and 1973.
4. P. Hill, *Two problems of Fuchs concerning tor and hom*, J. Algebra **19** (1971), 379–383.
5. R. Hunter, F. Richman, and E. Walker, *Warfield modules*, Lecture Notes in Math., vol. 616, Springer-Verlag, Berlin and New York, 1977, pp. 87–123.
6. P. Keef, *Some isomorphisms of abelian groups involving the Tor functor*, Proc. Amer. Math. Soc. **110** (1990), 27–37.

Department of Mathematics, Whitman College, Walla Walla, Washington 99362
E-mail address: KEEF@WHITMAN.BITNET