PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 118, Number 3, July 1993

MEASURES OF NONCOMPACTNESS AND UPPER SEMI-FREDHOLM PERTURBATION THEOREMS

FERNANDO GALAZ-FONTES

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. We introduce the concept of a perturbation function, which allows us to give a general approach to the question of obtaining perturbation theorems for upper semi-Fredholm operators. Also, we show that the usual measures of noncompactness of continuous linear operators, as well as other related quantities, are perturbation functions.

1. INTRODUCTION

Several ways of measuring the noncompactness of a continuous linear operator T acting from a Banach space X into another Banach space Y, for short $T \in \mathscr{L}(X, Y)$, have been considered. We have the quotient norm $\|\cdot\|_K$ induced by the subspace of compact linear operators $K(X, Y)$ (Example 1), the Kuratowski measure of noncompactness (Example 2), the ball measure of noncompactness (Example 3), the measure of noncompactness studied by Lebow and Schechter (Example 4), and the characteristics Δ and τ introduced by Schechter (Examples 5 and 6). One of the main applications of these quantities is in providing perturbation results for semi-Fredholm operators [10–12].

In this paper, by means of what we have called a perturbation function, we present a general approach to the question of obtaining perturbation theorems for upper semi-Fredholm operators $(T \in \Phi_+(X, Y))$, these are operators $T \in \mathscr{L}(X, Y)$ with finite-dimensional null space $N(T)$ and closed range $R(T)$. To this end we construct, for each perturbation function, characteristics Δ and Γ, which are similar to those studied by Schechter [9]. Also, we establish that the quantities mentioned above are perturbation functions.

Let $T \in \Phi(X, Y)$, that is, $T \in \mathscr{L}(X, Y)$ and both $n(T) = \dim N(T)$ and $d(T) = \dim Y/R(T)$ are finite. Making $\mathscr{L}(X) = \mathscr{L}(X, X)$ and $\Phi(X) = \Phi(X, X)$, recall that the essential spectral radius of $T \in \mathscr{L}(X)$ is

$$r_e(T) = \max\{|\lambda| : T - \lambda I \notin \Phi(X)\}$$

and that we have [2, §I.4]

$$r_e(T) = \lim_{n\to\infty} \|T^n\|_K^{1/n}. \tag{1.1}$$

Received by the editors November 22, 1991.
1991 *Mathematics Subject Classification*. Primary 47A55, 47A53.

Our method shows that (1.1) is valid if instead of the quotient norm $\|\cdot\|_K$ we employ any perturbation function. This gives, for the various examples we have considered, a unified and simple approach for obtaining such a formula.

We have also been able to establish a formula for the upper semi-Fredholm radius

$$r_s(T) = \min\{|\lambda| : T - \lambda I \notin \Phi_+(X)\}, \tag{1.2}$$

of $T \in \Phi_+(X) = \Phi_+(X, X)$. This formula is based on the corresponding results proved independently by Zemánek [12] and Tylli [11].

2. Perturbation functions and measures of noncompactness

Throughout this work X, Y, and Z will always denote complex Banach spaces, while M, V, and W will be (closed linear) subspaces of X, each of them having infinite dimension. The natural embedding of M into X will be indicated by J_M, and $T|_M \in \mathscr{L}(M, Y)$ will be the restriction of $T \in \mathscr{L}(X, Y)$ to M.

The minimum modulus of $T \in \mathscr{L}(X, Y)$ will be denoted by $j(T)$, that is,

$$j(T) = \inf\{\|Tx\|/\|x\| : x \in X\}.$$

Notice that $j(T) > 0$ if and only if T is one-to-one and has closed range. If this is the case, then $T^{-1} \in \mathscr{L}(R(T), X)$ and $j(T) = \|T^{-1}\|^{-1}$.

Definition 1. We define a perturbation function to be a function γ, assigning to each pair of complex Banach spaces X, Y, and $T \in \mathscr{L}(X, Y)$ a nonnegative number $\gamma(T)$, with the following properties:

(a) $\gamma(\lambda T) = |\lambda|\gamma(T)$, $\lambda \in \mathbb{C}$.
(b) $\gamma(T + K) = \gamma(T)$, $K \in K(X, Y)$.
(c) $\gamma(T) \le \|T\|$.
(d) $j(T) \le \gamma(T)$.
(e) $\gamma(T|_M) \le \gamma(T)$.

Next we show that the usual measures of noncompactness, and other related quantities, are perturbation functions.

Example 1. The quotient norm induced by $K(X, Y)$ is

$$\|T\|_K = \inf\{\|T - K\| : K \in K(X, Y)\}, \qquad T \in \mathscr{L}(X, Y).$$

Clearly $\|\cdot\|_K$ has properties (a)–(c) and (e). To verify property (d), let us take $T \in \mathscr{L}(X, Y)$. Given $K \in K(X, Y)$ and $\varepsilon > 0$, choose $M \subset X$ having finite codimension and such that $\|Kx\| \le \varepsilon\|x\|$, $x \in M$ [4, Theorem III.2.3]. Thus

$$\|Tx - Kx\| \ge \|Tx\| - \|Kx\| \ge (j(T) - \varepsilon)\|x\|, \qquad x \in M.$$

Making now $\varepsilon \to 0$, we find $\|T - K\| \ge j(T)$. Hence $\|T\|_K \ge j(T)$.

Example 2. If A is a bounded subset of X, its Kuratowski measure of noncompactness is given by

$$\Psi_X(A) = \inf\left\{d > 0 : A \subset \bigcup_{k=1}^{n} D_k, \text{ where each } D_k \text{ has diameter } \le d\right\}.$$

Then, the Kuratowski measure of noncompactness of $T \in \mathscr{L}(X, Y)$ is

$$\gamma(T) = \sup\{\Psi_Y(TA)/\Psi_X(A) : \Psi_X(A) \neq 0\}.$$

It is well known, and easy to verify, that γ has properties (a)–(c) and (e) [2, Lemma I.2.8]. Assume $T \in \mathscr{L}(X, Y)$ and $j(T) > 0$. Take $A \subset X$ with $\Psi_X(A) > 0$. Let us suppose that $T(A) \subset \bigcup_{k=1}^n D_k$, where each D_k has diameter $\leq d$. Then $A \subset \bigcup_{k=1}^n T^{-1}D_k$ and diameter $T^{-1}D_k \leq j(T)^{-1}d$. It follows that $\Psi_X(A) \leq j(T)^{-1}d$. This implies $\Psi_Y(TA)/\Psi_X(A) \geq j(T)$; therefore, $\gamma(T) \geq j(T)$.

Example 3. The ball measure of noncompactness of a bounded set $A \subset X$ is

$$\widetilde{\Psi}_X(A) = \inf\left\{r > 0 : A \subset \bigcup_{k=1}^n B(x_k, r)\right\},$$

where $B(x_k, r) = \{x \in X : \|x - x_k\| \leq r\}$. Then, denoting by B_X the closed unit ball in X, the ball measure of noncompactness of $T \in \mathscr{L}(X, Y)$ is

$$\tilde{\gamma}(T) = \widetilde{\Psi}_Y(TB_X).$$

As in the previous examples, properties (a)–(c) and (e) are well known and easy to verify [2, Lemma I.2.8].

To establish property (d) we first prove that $\widetilde{\Psi}_X(B_M) = 1$ for any infinite-dimensional subspace M of X. Take $r = \widetilde{\Psi}_X(B_M)$ and $\varepsilon > 0$. Then $B_M \subset \bigcup_{k=1}^n B(y_k, r+\varepsilon)$, for some $y_1, \dots, y_n$ in X. Noting that the linear span of these vectors is finite dimensional, we have $r+\varepsilon \geq 1$ [4, Lemma V.1.1]. Letting $\varepsilon \to 0$ we obtain $r \geq 1$, so $\widetilde{\Psi}_X(B_M) \geq 1$. Since $\widetilde{\Psi}_X(B_m) \leq 1$, the conclusion is clear.

Assume $T \in \mathscr{L}(X, Y)$ and $j(T) > 0$. This implies $j(T)B_W \subset TB_X$, where $W = R(T)$. Thus, by what we have just proved,

$$\widetilde{\Psi}_Y(TB_X) \geq j(T)\widetilde{\Psi}_Y(B_W) = j(T).$$

Example 4. Let us define

$$m(T) = \inf\{\|T|_M\| : \operatorname{codim} M < \infty\}, \qquad T \in \mathscr{L}(X, Y).$$

It is well known that m has properties (a)–(c) [2, Corollary I.2.22]; property (d) is clear. Property (e) is obtained by noting that if V has finite codimension in X, then $V \cap M$ has finite codimension in M.

Example 5. Let us define

$$\Delta(T) = \sup_{M \subset X} \inf\{\|T|_V\| : V \subset M\}.$$

Properties (a)–(c) are established in [9], while properties (d) and (e) follow easily from the definition.

Example 6. Let us define $\tau(T) = \sup\{j(T|_M) : M \subset X\}$.

We will prove property (b), the other properties are clear. Let $K \in K(X, Y)$. Given $\varepsilon > 0$, choose as V a subspace of X having finite codimension, and where $\|Kx\| \leq \varepsilon\|x\|$, $x \in V$. Take M to be an infinite-dimensional subspace

of X. Hence $\dim V \cap M = \infty$, and

$$\|Tx\| \leq \|Tx - Kx\| + \varepsilon\|x\|, \qquad x \in V \cap M.$$

Thus

$$j(T|_M) \leq j([T-K]|_{V\cap M}) + \varepsilon \leq \tau(T-K) + \varepsilon.$$

It follows that $\tau(T) \leq \tau(T-K)$. This in turn implies our conclusion.

The next lemma follows directly from Definition 1.

Lemma 1. *If γ is a perturbation function, then the following hold.*

$$\text{(2.1)} \qquad \begin{aligned} &\text{(i)} \quad \gamma(K) = 0,\ K \in K(X, Y). \\ &\text{(ii)} \quad \gamma(J_M) = 1. \end{aligned}$$

Remark 1. We want to notice that the measures of noncompactness given in Examples 1–5 are in fact submultiplicative seminorms:

$$\gamma(T+S) \leq \gamma(T) + \gamma(S), \qquad T, S \in \mathscr{L}(X, Y);$$
$$\gamma(ST) \leq \gamma(S)\gamma(T), \qquad T \in \mathscr{L}(X, Y),\ S \in \mathscr{L}(Y, Z).$$

The measures of noncompactness indicated in Examples 2–4 are equivalent [2, Lemma I.2.8, Theorem I.2.21]; however, Astala and Tylli [1] showed that this does not happen with $\|\cdot\|_K$ and $\tilde{\gamma}(T)$. Since $\Delta(T)$ annihilates precisely on the strictly singular operators [9] and $\tilde{\gamma}$ on the compact operators, it follows that Δ and $\tilde{\gamma}$ are not equivalent.

3. Perturbation theorems

The following result, which is basic in our development, is due to Schechter [8], for whose proof we refer.

Lemma 2. *Let $T \in \mathscr{L}(X, Y)$. Then $T \notin \Phi_+(X, Y)$ if and only if there is an infinite-dimensional (closed linear) subspace $M \subset X$ and $K \in K(X, Y)$ such that $T = K$ on M.*

Before stating our first perturbation theorem, let us recall that the index of an upper semi-Fredholm operator T is $\kappa(T) = n(T) - d(T)$. Also, in all our statements we will always assume that γ is a perturbation function.

Lemma 3. *If $P \in \mathscr{L}(X)$ and $\gamma(P) < 1$, then $I + P \in \Phi(X)$ and $\kappa(I+P) = 0$.*

Proof. First we show that $I + P \in \Phi_+(X)$. Suppose this is not so and take M and K as in Lemma 2. Then $J_M = -P|_M + K|_M$. Thus, by (2.1) and the properties of γ, we have

$$1 = \gamma(J_M) = \gamma(P|_M) \leq \gamma(P) < 1.$$

This contradiction shows that $I + P \in \Phi_+(X)$.

Let $0 \leq \lambda \leq 1$. Then $\gamma(\lambda P) < 1$ and so, by what we have just proved, $I + \lambda P \in \Phi_+(X)$. Thus, by the continuity of the index, $\kappa(I+P) = \kappa(I) = 0$. Therefore $I + P \in \Phi(X)$.

Lemma 4. *If $T \in \mathscr{L}(X)$, then $r_e(T) \leq \gamma(T) \leq \|T\|_K$.*

Proof. Since

$$\gamma(T) = \gamma(T+K) \leq \|T+K\|, \qquad K \in K(X, Y),$$

we have $\gamma(T) \leq \|T\|_K$.

Take $\lambda \in \mathbb{C}$, $|\lambda| > \gamma(T)$. Then $\gamma(T/\lambda) < 1$ and thus, by Lemma 3, $T - \lambda I \in \Phi(X)$. It follows that $\gamma(T) \geq r_e(T)$.

Theorem 5. $r_e(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}$, $T \in \mathscr{L}(X)$.

Proof. From Lemma 4 we have

$$r_e(T^n) \leq \gamma(T^n) \leq \|T^n\|_K. \tag{3.1}$$

Since $r_e(T^n) = r_e(T)^n$, the conclusion follows from (1.1) and (3.1).

Remark 2. Theorem 5 was established for the ball measure of noncompactness by Gohberg, Goldenstein, and Markus [3], for the Kuratowski measure of noncompactness by Nussbaum [7], for the measure of noncompactness given in Example 4 by Lebow and Schechter [6], and for the quantities Δ and τ by Schechter [9].

Lemma 3 is a perturbation result for the identity $I \in \mathscr{L}(X)$. To obtain perturbation results for any $T \in \Phi_+(X, Y)$, we will now define other quantities.

Definition 2. Given a measure of noncompactness γ, we make the following definitions for $T \in \mathscr{L}(X, Y)$:

$$\Gamma_M(T) = \inf\{\gamma(T|_V) \colon V \subset M\}, \qquad \Gamma(T) = \Gamma_X(T);$$
$$\Delta_M(T) = \sup\{\Gamma_V(T) \colon V \subset M\}, \qquad \Delta(T) = \Delta_X(T).$$

Remark 3. The quantities Γ and Δ were introduced, respectively, by Gramsch [5] and Schechter [9] in connection with the measure of noncompactness considered in Example 4, although this was not done explicitly.

Theorem 6. *Let $P, T \in \mathscr{L}(X, Y)$. If $\Delta(P) < \Gamma(T)$, then T and $T+P$ belong to $\Phi_+(X, Y)$ and $\kappa(T) = \kappa(T+P)$.*

Proof. First we will prove that $T+P \in \Phi_+(X, Y)$. Suppose this is not so. Then, by Lemma 2, there is a subspace M and $K \in K(X, Y)$ such that $T + P = K$ on M. Therefore

$$\Gamma(T) \leq \Gamma_M(T) = \Gamma_M(P) \leq \Delta(P);$$

this contradicts the hypothesis.

Let $0 \leq \lambda \leq 1$. Then $\Delta(\lambda P) < \Gamma(T)$ and so, by what we have just proved, $T + \lambda P \in \Phi_+(X, Y)$. Therefore $T \in \Phi_+(X, Y)$ and, since the index is a continuous function, $\kappa(T) = \kappa(T+P)$.

Theorem 7. *Let $T \in \mathscr{L}(X, Y)$. Then $T \in \Phi_+(X, Y)$ if and only if $\Gamma(T) > 0$.*

Proof. If $\Gamma(T) > 0$, by taking $P = 0$ in Theorem 6 we find $T \in \Phi_+(X, Y)$.

Assume that $T \in \Phi_+(X, Y)$. Express X as a direct sum $X = N(T) \oplus M$. Note that the subspace M has finite codimension and $j(T|_M) > 0$. Take V to be an infinite-dimensional subspace of X. Thus $W = M \cap V$ is also infinite dimensional and

$$j(T|_M) \leq j(T|_W) \leq \gamma(T|_W) \leq \gamma(T|_V).$$

Hence $\Gamma(T) > 0$.

Given a perturbation function γ, it is easy to establish that $\Delta = \Delta(\gamma)$ is also a perturbation function satisfying

$$\Delta(T) \leq \gamma(T), \qquad T \in \mathscr{L}(X, Y).$$

Consider now $\Delta(\Delta(\gamma))$. The next lemma implies that $\Delta(\gamma) = \Delta(\Delta(\gamma))$.

Lemma 8. $\Gamma_M(T) = \inf\{\Delta_V(T) \colon V \subset M\}$.

Proof. Let $c = \inf\{\Delta_V(T) \colon V \subset M\}$. If V is a subspace of M we have $\Delta_V(T) \geq \Gamma_V(T) \geq \Gamma_M(T)$. Thus $c \geq \Gamma_M(T)$.

Given $\varepsilon > 0$, there is a subspace V such that $\gamma(T|_V) \leq \Gamma_M(T) + \varepsilon$. Since $\Delta_V(T) \leq \gamma(T|_V)$, it follows that $c \leq \Gamma_M(T) + \varepsilon$. Letting $\varepsilon \to 0$, we obtain $c \leq \Gamma_M(T)$. The conclusion is now clear.

Let us recall that $P \in \mathscr{L}(X, Y)$ is said to be strictly singular if $j(T|_M) = 0$ for each infinite-dimensional subspace M of X.

Proposition 9. *If $P \in \mathscr{L}(X, Y)$ is strictly singular, then $\Delta(P) = 0$.*

Proof. Let V be an infinite-dimensional subspace of X. Given $\varepsilon > 0$, there is an infinite-dimensional subspace M of V, such that $\|T|_M\| \leq \varepsilon$ [4, Theorem III.1.9]. Thus $\Gamma_V(T) \leq \gamma(T|_M) \leq \varepsilon$. It follows that $\Gamma_V(T) = 0$ and so $\Delta(P) = 0$.

Definition 3. For $T \in \Phi_+(X, Y)$ and M a subspace of X, let T_M denote $T \colon M \to T(M)$. If $P \in \mathscr{L}(X, Y)$, we now define

$$K(P, T) = \sup\{\Delta(P \circ T_M^{-1}) \colon M \subset X \text{ has finite codimension, } j(T_M) > 0\}.$$

Theorem 10. *Let $T \in \Phi_+(X, Y)$ and $P \in \mathscr{L}(X, Y)$. If $K(P, T) < 1$, then $T + P \in \Phi_+(X, Y)$ and $\kappa(T + P) = \kappa(T)$.*

Proof. Let M be a subspace of X having finite codimension and such that $\Delta(P \circ T_M^{-1}) < 1$. Since $\Gamma(J_{TM}) = 1$, it follows from Theorem 6 that $J_{T(M)} + P \circ T_M^{-1} \in \Phi_+(T(M), X)$. Now $T_M \in \Phi_+(M, T(M))$ implies

$$T|_M + P = (J_{T(M)} + P \circ T_M^{-1}) \circ T_M \in \Phi_+(M, X).$$

Hence, noticing that $T|_M + J_M$ is the restriction of $T + I$ to M and that M has finite codimension, we conclude that $I + T \in \Phi_+(X)$.

Let $0 < \lambda \leq 1$. Then $K(\lambda P, T) = \lambda K(P, T) < 1$. Thus, by what we have just proved, $T + \lambda I \in \Phi_+(X)$, $0 \leq \lambda \leq 1$. $\kappa(T) = \kappa(T + I)$ follows now from the continuity of the index.

Remark 4. In the terminology of [10], Theorems 6 and 10 indicate that $F(P, T) = \Delta(P)/\Gamma(T)$ and $K(P, T)$ are perturbation functions for $\Phi_+(X, Y)$.

Zemánek [12] proved that the semi-Fredholm radius satisfies

$$r_s(T) = \lim_{n \to \infty} \nu(T^n)^{1/n}, \qquad T \in \mathscr{L}(X), \tag{3.2}$$

where

$$\nu(T) = \sup\{j(T|_M) \colon \operatorname{codim} M < \infty\}.$$

We will now establish a similar formula in our abstract context.

The next lemma follows directly from the proof of Theorem 7.

Lemma 11. $\Gamma(T) \geq \nu(T)$, $T \in \mathscr{L}(X, Y)$.

Since $\Delta(I) = 1$, by Theorem 6, we obtain the following result.

Lemma 12. *Assume* $T \in \Phi_+(X)$. *If* $1 < \Gamma(T)$, *then* $I + T \in \Phi_+(X, Y)$ *and* $\kappa(T) = \kappa(T + I)$.

Theorem 13. $r_s(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}$, $T \in \Phi_+(X)$.

Proof. From Lemma 11 we have $\Gamma(T^n) \geq \nu(T^n)$. Thus, by (3.2),

$$\liminf_{n\to\infty} \Gamma(T^n)^{1/n} \geq r_s(T).$$

Now assume $0 < |\lambda| < \limsup_{n\to\infty} \Gamma(T^n)^{1/n}$. Choose $n \in \mathbb{N}$ such that $|\lambda|^n < \Gamma(T^n)$. Then $\Gamma(T^n/\lambda^n) > 1$ and so we can use Lemma 12 to obtain $T^n - \lambda^n I \in \Phi_+(X)$. Since $T^n - \lambda^n I = S \circ (T - \lambda I)$ for some $S \in \mathscr{L}(X)$, it follows that $T - \lambda I \in \Phi_+(X)$. This shows

$$r_s(T) \geq \limsup_{n\to\infty} \Gamma(T^n)^{1/n}.$$

The conclusion is now clear.

References

1. K. Astala and H. O. Tylli, *On the bounded compact approximation property and measures of non-compactness*, J. Funct. Anal. **70** (1987), 388–401.
2. D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Clarendon Press, Oxford, 1987.
3. I. T. Gohberg, L. S. Goldenstein, and A. S. Markus, *Investigation of some properties of bounded linear operators in connection with their q-norms*, Uchen. Zap. Kishinev Gos. Univ. **29** (1957), 29–36. (Russian)
4. S. Goldberg, *Unbounded linear operators*, McGraw-Hill, New York, 1966.
5. B. Gramsch, *Über analytische störungen und den Index von Fredholm operatoren auf Banachraumen*, Dept. of Mathematics, Univ. Maryland, 1969.
6. A. Lebow and M. Schechter, *Semigroups of operators and measures of non-compactness*, J. Funct. Anal. **7** (1971), 1–26.
7. R. Nussbaum, *The radius of the essential spectrum*, Duke Math. J. **37** (1970), 473–478.
8. M. Schechter, *Riesz operators and Fredholm perturbations*, Bull. Amer. Math. Soc. **64** (1968), 1139–1144.
9. ——, *Quantities related to strictly singular operators*, Indiana Univ. Math. J. **21** (1972), 1061–1071.
10. M. Schechter and R. Whitley, *Best Fredholm perturbation theorems*, Studia Math. **90** (1988), 175–190.
11. H. O. Tylli, *On the asymptotic behaviour of some quantities related to semi-Fredholm operators*, J. London Math. Soc. (2) **31** (1985), 340–348.
12. J. Zemánek, *Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour*, Studia Math. **80** (1984), 219–234.

Centro de Investigación en Matemáticas, Apdo postal 402, 36 000 Guanajuato, Mexico
E-mail address: cimat@unamvm1.bitnet