A COUNTEREXAMPLE TO THE INFINITY VERSION OF THE HYERS AND ULAM STABILITY THEOREM

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(Communicated by Andrew M. Bruckner)

Abstract. Hyers and Ulam proved a stability result for convex functions, defined in a subset of $\mathbb{R}^n$. Here we give an example showing that their result cannot be extended to those functions defined in infinite-dimensional normed spaces. Also, we give a positive result for a particular class of approximately convex functions, defined in a Banach space, whose norm satisfies the so-called convex approximation property.

1. Introduction

In this paper we discuss the following problem: let $\Delta$ be a convex subset of a Banach space $X$. Consider an arbitrary $\epsilon$-convex function $f: \Delta \to \mathbb{R}$, that is, a function that satisfies for every $x, y \in \Delta$ and every $\lambda \in [0, 1]$ the inequality:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \epsilon.$$

Is it true that there exists a convex function $g: \Delta \to \mathbb{R}$ such that $|g(x) - f(x)| \leq K\epsilon \forall x \in \Delta$, where $K$ is a constant depending only on $X$? A positive answer was given by Hyers and Ulam [HU] (see also [Gr, C]) in the case $X = \mathbb{R}^n$ (with any norm!); the best known estimates are with $K_n = \min(M_n, L_n)$ where $M_n = (n^2 + 3n)/(4n + 4)$ and $L_n = m/2$ for $2^{m-1} \leq n < 2^m$ (see also [Ge] for a discussion concerning these constants and related questions). We will show that in the infinite-dimensional case the stability theorem of [HU] does not hold. Also we shall give a positive answer when the space $X$ and the functions satisfy some additional properties. For midpoint convex functions a counterexample is known [Ge]; for a positive result concerning $\eta$-approximately convex functions, which are also $\epsilon$-subadditive, see [K].

2. Counterexample

Let $X$ be a Banach space. We denote by $B(X)$ (resp. $B_r(X)$) the unit ball (resp. ball of radius $r$) of $X$, i.e., $B(X) = \{x \in X : \|x\| \leq 1\}$ (resp.
$B_r(X) = \{ x \in X : \| x \| \leq r \}$, and if $A$ is a subset of $X$, by $\text{co} \, A$ its convex hull.

We say that the set $A \subseteq X$ satisfies the condition $C^0(\epsilon)$ if $x, y \in A$ implies $(x + y)/2 \in A + B_r(X)$. We say that the set $A$ satisfies the condition $c^0(\epsilon)$ if $x, y \in A$ and $\lambda \in [0, 1]$ implies $\lambda x + (1 - \lambda)y \in A + B_r(X)$.

Let $l_1$ be the Banach space of absolutely convergent sequences with the usual norm (that we will indicate by $\| x \|_1$). We will denote by $\{ e_i \}$ the standard basis of $l_1$. Let $C^+ = \{ x \in l_1 : x_i \geq 0 \}$, the positive cone of $l_1$, and $B^+ = C^+ \cap B(l_1)$.

For $p \in (0, 1)$ set $S_p = \{ x \in B^+ : \sum_{i=1}^{+\infty} x_i^p \leq 1 \}$. We need the following two propositions: the first one is in [L], but we give the proof since the paper is not very accessible, the second is in [CP].

**Proposition 1.** The set $S_p$ satisfies condition $C^0(2^{1/p} - 1)$.

**Proof.** Let $x, y \in S_p$ such that $\sum_{i=1}^{+\infty} x_i^p = \sum_{i=1}^{+\infty} y_i^p = 1$. Then we have

$$\inf_{z \in S_p} \left\| \frac{x + y}{2} - z \right\|_1 = \inf_{z \in S_p} \sum_{i=1}^{+\infty} \left( x_i + y_i \right) - 2z_i \right\|_1 = \inf_{z \in S_p} \sum_{i=1}^{+\infty} \left( x_i^p + y_i^p \right) - 2z_i \right\|_1 \leq \inf_{z \in S_p} \sum_{i=1}^{+\infty} \left( x_i^p + y_i^p \right) - 2z_i \right\|_1.$$

For positive numbers, $s^p + t^p = z^p$ implies $s + t \leq z$; thus

$$\inf_{z \in S_p} \left\| \frac{x + y}{2} - z \right\|_1 \leq \inf_{z \in S_p} \left\| \frac{x' + y'}{2} - z \right\|_1.$$

Now let $x$ and $y$ be in $S_p$; define $x'$, $y'$ (still in $S_p$ and with disjoint supports) as follows:

$$x'_{2n} = 0, \quad x'_{2n-1} = x_n, \quad y'_{2n-1} = y_n, \quad y'_{2n} = y_n.$$

Then, by using (2.1) and (2.2), we obtain

$$\inf_{z \in S_p} \left\| \frac{x + y}{2} - z \right\|_1 \leq \inf_{z \in S_p} \left\| \frac{x' + y'}{2} - z \right\|_1,$$

so we can suppose that $x$ and $y$ have disjoint supports. With this restriction, letting $z_i = (x_i + y_i)/2^{1/p}$ we obtain

$$\inf_{z \in S_p} \left\| \frac{x + y}{2} - z \right\|_1 \leq \left( \frac{1}{2} - \frac{1}{2^{1/p}} \right) \sum_{i=1}^{+\infty} (x_i + y_i) \leq \left( \frac{1}{2} - \frac{1}{2^{1/p}} \right) \left( \sum_{i=1}^{+\infty} (x_i + y_i)^p \right)^{1/p} = 2^{1/p - 1} - 1.$$

Since $x$ and $y$ are arbitrary, $S_p$ satisfies condition $C^0(2^{1/p} - 1)$.

**Proposition 2** (see [CP]). If $A$ satisfies $C^0(\epsilon)$ then $A$ satisfies $c^0(2\epsilon)$.

So, in particular, $S_p$ satisfies condition $c^0(2^{1/p} - 2)$.

Now let us give the counterexample. For each $p \in (0, 1)$, let $f_p : B^+ \subset l^1 \rightarrow \mathbb{R}$ be defined in the following way: $f_p(x) = \text{dist}(x, S_p) = \inf_{z \in S_p} \| x - z \|_1$. First of all we will prove that $f_p$ is a $(2^{1/p} - 2)$-convex function. Take $\eta > 0$,
We take \( x, y \in B^+ \), and \( z_1, z_2 \in S_p \) such that \( \|x - z_1\|_1 \leq f_p(x) + \eta \) and \( \|y - z_2\|_1 \leq f_p(y) + \eta \); then, by using Proposition 2, we obtain

\[
f_p(\lambda x + (1 - \lambda)y) = \inf_{z \in S_p} \|\lambda x + (1 - \lambda)y - z\|_1
\]

\[
\leq \inf_{z \in S_p} (\|\lambda x - z_1\|_1 + (1 - \lambda)\|y - z_2\|_1
\]

\[
+ \|\lambda z_1 + (1 - \lambda)z_2 - z\|_1)
\]

\[
\leq \lambda f_p(x) + (1 - \lambda)f_p(y) + \eta + \inf_{z \in S_p} \|\lambda z_1 + (1 - \lambda)z_2 - z\|_1
\]

\[
\leq \lambda f_p(x) + (1 - \lambda)f_p(y) + \eta + 2^{1/p} - 2.
\]

The conclusion follows from the arbitrariness of \( \eta \).

Now suppose that there exists \( K \) such that there exists a convex function \( g: B^+ \to \mathbb{R} \) satisfying \( |f_p(x) - g(x)| < K(2^{1/p} - 2) \). Take \( \overline{p} \in (0, 1) \) such that \( K(2^{1/\overline{p}} - 2) \leq 1/4 \). Let

\[
x_n = (1/n, \ldots, 1/n, 0, 0, \ldots);
\]

we have (since \( f_p(e_1) = 0 \))

\[
f_p(x_n) \leq \frac{1}{4} + g(x_n) \leq \frac{1}{4} + \sum_{i=1}^{n} \frac{g(e_i)}{n} \leq \frac{1}{4} + \sum_{i=1}^{n} \frac{f_p(e_i) + 1/4}{n} = \frac{1}{2}.
\]

This is a contradiction, since an easy calculation shows that \( f_{\overline{p}}(x_n) = 1 - n^{1-1/\overline{p}} \to 1 \) as \( n \to \infty \).

**Remark 1.** The finite-dimensional version of this counterexample shows that, at least asymptotically, the constants \( K_n \) that appear in [C] are best possible. In fact, define \( H_n \) as the best constant such that for every convex subset \( \Delta \) of \( \mathbb{R}^n \) and for every \( \varepsilon \)-convex function \( f \) on \( \Delta \) there exists a convex function \( g \) on \( \Delta \) such that \( |g(x) - f(x)| \leq H_n \varepsilon \) for every \( x \in \Delta \). Consider \( S_p^n = \{ x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^{n} x_i^p \leq 1 \} \), and define, as before, \( f_p(x) = \inf_{z \in S_p^n} \|x - z\|_1 \). Then, if \( x_n = (1/n, \ldots, 1/n) \), proceeding in the same way as before, we obtain

\[
1 - n^{1-1/p} = f_p(x_n) \leq H_n(2^{1/p} - 2) + g(x_n)
\]

\[
\leq H_n(2^{1/p} - 2) + \sum_{i=1}^{n} \frac{g(e_i)}{n}
\]

\[
\leq H_n(2^{1/p} - 2) + \sum_{i=1}^{n} \frac{f_p(e_i) + H_n(2^{1/p} - 2)}{n}
\]

\[
= 2H_n(2^{1/p} - 2).
\]

This implies

\[
H_n \geq \frac{1 - n^{1-1/p}}{4(2^{1/p} - 1 - 1)} \sim \frac{\log_2 n}{4} \quad \text{as } p \to 1^{-}.
\]

**Remark 2.** Now consider the Hilbert space \( l_2 = \{ x \} : \sum_{n=1}^{+\infty} x_n^2 < +\infty \} \) with the usual norm. Then the previous example works, also, if one thinks of \( B^+ \) as a subset of \( l_2 \). Notice that the convex set \( B^+ \) is a convex subset of \( l_2 \) (since algebraically \( l_1 \subset l_2 \)) and, also, it is a closed subset in \( l_2 \). Take, in fact, a
sequence \( \{x_n\} \subset B^+ \) such that \( x_n \to x \) in the \( l_2 \) norm. Then \( \{x_n\} \) converges to \( x \) in the weak-topology of \( l_2 \) and so it converges coordinatewise. But, also, as a bounded sequence in \( l_1 \) it has a subnet converging in the weak*-topology of \( l_1 \) (and in particular coordinatewise) to an element of \( l_1 \). So \( x \) belongs to \( l_1 \), thus to \( B^+ \).

**Remark 3.** As the previous remark pointed out, the stability problem is (in some sense) independent of the norm-topology of the Banach space in which the domain of our \( \epsilon \)-convex functions lie. A way to relate the norm to the functions is to ask that they must satisfy some lipschitz condition. Notice that the functions \( f_p \) considered in our counterexample are 1-lip, since it follows easily, from the definition of distance, that \( |f_p(x) - f_p(y)| \leq \|x - y\|_1 \). The same condition is not satisfied if we think of our example embedded in \( l_2 \). In fact, as a simple calculation shows, for any fixed \( h \) our functions are not \( h \)-lip, for every \( p \), in the \( l_2 \)-norm. This will follow directly from Theorem 1 of the next section.

### 3. A POSITIVE RESULT

In this section, we prove that the construction of our counterexample is possible since \( l_1 \) lacks a convexity property called the "convex approximation property" (C.A.P. for short). We will say that a Banach space \( X \) has C.A.P. if for every \( \epsilon > 0 \) and \( r > 0 \), there exists an integer \( p = p(\epsilon, r) \) such that for every \( A \subset B_r(X) \) we have \( \text{co} A \subset \text{co}_p A + B_r(X) \), where \( \text{co}_p A = \{ x \in X : x = \sum_{i=0}^p \alpha_i x_i, \ x_i \in X, \ \alpha_i \geq 0, \ \sum_{i=0}^p \alpha_i = 1 \} \). In other words, each element of \( \text{co} A \) can be \( \epsilon \)-approximated by a convex combination of no more than \( p \) vectors of \( A \). We will say that \( X \) is B-convex [P] if there exist constants \( c > 0 \), \( p > 1 \) such that for every \( n \) and all independent random variables \( g_1, \ldots, g_n \) with values in \( X \) we have

\[
E \left( \left\| \sum_{i=1}^n g_i \right\|_X^p \right) \leq c^p \sum_{i=1}^n E \|g_i\|_X^p .
\]

It is easy to construct a sequence of independent random variables with values in \( l_1 \) such that (3.1) does not hold. This implies that \( l_1 \) is not B-convex and so it does not have C.A.P. since we have the following result of Bruck [B].

**Proposition 3.** A Banach space \( X \) has C.A.P. if and only if it is B-convex. Moreover, in this case, there exist constants \( c > 0 \), \( q > 1 \) depending on \( X \) and \( r \) such that, for every \( A \subset B_r(X) \), we can choose \( p \) (in the definition of C.A.P.) via \( p \leq c \epsilon^{q/(1-q)} \).

The main result is the following:

**Theorem 1.** Let \( X \) a B-convex Banach space, \( h > 0 \), and \( \Delta \) a bounded convex subset of \( X \). Then, for every \( f : \Delta \to \mathcal{R} \), which is \( \epsilon \)-convex and \( h \)-lip, there exists (for \( \epsilon \) sufficiently small) a constant \( K \) (depending on \( X \), \( h \), and \( \text{diam}(\Delta) \)) such that there exists a convex function \( g : \Delta \to \mathcal{R} \) satisfying \( |f(x) - g(x)| \leq Ke \log_2 \epsilon \).

We will use the following lemma, the proof of which can be found in several papers and in [C] with the best known constants.
Lemma 1. Let $X$ be a $B$-convex Banach space and $f: \Delta \subset X \rightarrow \mathbb{R}$ an $\epsilon$-convex function. Then for $x_0, \ldots, x_p \in \Delta$, $\alpha_0, \ldots, \alpha_p \in [0, 1]$, $\alpha_1 + \cdots + \alpha_p = 1$, and all integers $p$, we have $f(\sum_{i=0}^{p} \alpha_i x_i) \leq \sum_{i=0}^{p} \alpha_i f(x_i) + 2K_p \epsilon$. (For the definition of $K_n$ see the introduction.)

Proof of Theorem 1. Suppose that $\Delta \subseteq B_r(x)$ and let $d = \text{diam}(\Delta)$. We take $Z = X \oplus \mathbb{R}$ with $\|z\| = \|(x, y)\| = \|x\| + |y|$ $(x \in X, \ y \in \mathbb{R})$. It is easy to show that $Z$ is a $B$-convex Banach space since $X$ is $B$-convex, and so it has C.A.P. Since $f$ is $h$-lip then $f$ is bounded on $\Delta$. Let $m = \inf_{x \in \Delta} f(x)$, $M = \sup_{x \in \Delta} f(x)$, and $H_0 = \{z \in Z : z = (x, y), x \in \Delta, f(x) \leq y \leq M\}$. We have to show that $H_0$ is a bounded subset of $Z$. In fact, if we take $z_1, z_2 \in H_0$ we have

$$\|z_1 - z_2\| = \|x_1 - x_2\| + |y_1 - y_2| \leq d + M - m,$$

but as a consequence of the lipschitz condition on $f$ we have $M - m \leq h d$, so $\text{diam}(H_0) \leq (1 + h)d$.

Take $\eta > 0$. Then there exists an integer $p$ such that if $z \in \text{co} H_0$ there exist $z_0, \ldots, z_p \in H_0$ and $\alpha_0, \ldots, \alpha_p$ so that

$$\left\| z - \sum_{i=0}^{p} \alpha_i z_i \right\| = \left\| x - \sum_{i=0}^{p} \alpha_i x_i \right\| + \left\| y - \sum_{i=0}^{p} \alpha_i y_i \right\| \leq \eta.$$

Then by using Lemma 1

$$f(x) = f(x) - f\left(\sum_{i=0}^{p} \alpha_i x_i\right) + f\left(\sum_{i=0}^{p} \alpha_i x_i\right) \leq f(x) - f\left(\sum_{i=0}^{p} \alpha_i x_i\right) + \sum_{i=0}^{p} \alpha_i f(x_i) + 2K_p \epsilon \leq h \|x - \sum_{i=0}^{p} \alpha_i x_i\| + \sum_{i=0}^{p} \alpha_i y_i - y + y + 2K_p \epsilon \leq h' \eta + y + 2K_p \epsilon \quad (h' = \max(h, 1)).$$

Taking into account the upper bound for $p$ given in Proposition 3 and the value of the constant $K_n$, we obtain (for $p$ sufficiently large, that is, for $\eta$ small)

$$f(x) \leq h' \eta + (1 + \log_2 c \eta^{q/(1-q)}) \epsilon + y.$$ 

Since $\eta$ was arbitrary, we can choose: $\eta = q \epsilon/(1 - q) h' \log_2 \epsilon$ (this is the minimum point of the function $g(\eta) = h' \eta + (1 + \log_2 c \eta^{q/(1-q)}) \epsilon$). Then we obtain

$$f(x) \leq 2K \epsilon \log_2 \epsilon + y \quad (3.2)$$

for some negative constant $K$. Now define, for $x \in \Delta$, $g_0(x) = \inf\{y \in \mathbb{R} : (x, y) \in \text{co} H_0\}$. Then by (3.2) we have $f(x) - 2K \epsilon \log_2 \epsilon \leq g_0(x) \leq f(x)$. It is not hard to prove that $g_0$ is a convex function, and if we put $g(x) = g_0(x) + K \epsilon \log_2 \epsilon$ we obtain

$$f(x) - K \epsilon \log_2 \epsilon \leq g(x) \leq f(x) + K \epsilon \log_2 \epsilon,$$

which concludes the proof.
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