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AN INEQUALITY FOR SOME NONNORMAL OPERATORS—EXTENSION TO NORMAL APPROXIMATE EIGENVALUES

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ABSTRACT. An inequality of use in testing convergence of eigenvector calculations is extended. It is an improvement of Bernstein and Furuta's results for selfajoint operators and dominant operators respectively.

1. INTRODUCTION

In [5] one of the authors showed the following theorem, which is an improvement of Bernstein's in [2].

Theorem A. *If e is a unit eigenvector corresponding to an eigenvalue λ in a dominant operator A on a Hilbert space H, then*

$$|(g, e)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A-\lambda)g\|^2} \tag{1}$$

for all g in H for which $Ag \neq \lambda g$.

Here an operator A is called dominant if for each λ there is a real number $M_\lambda \geq 1$ such that $\|(A-\lambda)^*x\| \leq M_\lambda\|(A-\lambda)x\|$ for all x in H. We have to remark that $(A-\lambda)^*e = 0$ under the dominance of A. This says that λ is a normal eigenvalue of A, i.e., there is a nonzero vector x in H such that

$$(A-\lambda)x = 0 \quad \text{and} \quad (A-\lambda)^*x = 0. \tag{2}$$

In this note, we weaken the assumption of Theorem A to dominance of the operator to normality of the eigenvalue. More precisely,

Theorem 1. *If e is a unit eigenvector corresponding to a normal eigenvalue λ of A on a Hilbert space H, then* (1) *holds for all g in H for which $Ag \neq \lambda g$.*

Second, we give another generalization of Theorem A to normal approximate eigenvalues [3], i.e., a complex number λ is called a normal approximate eigenvalue of A if there exists a sequence $\{x_n\}$ of unit vectors such that

$$\|(A-\lambda)x_n\| \to 0 \quad \text{and} \quad \|(A-\lambda)^*x_n\| \to 0. \tag{3}$$

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It is known in [4] that the correspondence between the normal approximate eigenvalues and the normal eigenvalues is ensured by the Berberian representations in [1]. Therefore there is a bridge from Theorem 1 to this generalization, which is of course supported by the Berberian one.

2. Results

First of all, we cite the following lemma, which is one of the technical points in [2, 5].

Lemma 2. *Let A be an operator on a Hilbert space H. Then*

$$\|x\|^2\|Ax\|^2 - |(x, Ax)|^2 = \|x\|^2\|(A-\lambda)x\|^2 - |(x, (A-\lambda)x)|^2$$

for all x in H and complex numbers λ.

Proof of Theorem 1. Though the proof is almost the same as in [5], we give a proof for the sake of completeness. A given g in H is expressed as $g = \alpha e + f$, where $\alpha = (g, e)$ and $(e, f) = 0$. Put $B = A - \lambda$. Then $f \neq 0$, $Bg = Bf$, so

$$(g, Bg) = (g, Bf) = \alpha(e, Bf) + (f, Bf) = (f, Bf)$$

because $B^*e = 0$. Hence it follows from Lemma 1 that

$$\begin{aligned}\frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A-\lambda)g\|^2} &= \frac{(|\alpha|^2 + \|f\|^2)\|Bg\|^2 - |(g, Bg)|^2}{\|Bg\|^2} \\ &= |\alpha|^2 + \frac{\|f\|^2\|Bf\|^2 - |(f, Bf)|^2}{\|Bf\|^2} \geq |\alpha|^2.\end{aligned}$$

Remark. As in the above proof, the equality holds in (1) if and only if f is also an eigenvector of A.

Next we consider a generalization of Theorem 1 to normal approximate eigenvalues. Here we call the $\{e_n\}$ satisfying (3) a sequence of unit eigenvectors corresponding to a normal approximate eigenvalue λ of A.

Theorem 3. *If $\{e_n\}$ is a sequence of unit vectors corresponding to a normal approximate eigenvalue λ of A, then*

$$\overline{\lim}\,|(g, e_n)|^2 \leq \frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A-\lambda)g\|^2} \tag{4}$$

for all g in H for which $Ag \neq \lambda g$.

A proof of Theorem 3 will be presented in the next section.

3. The Berberian representation

We begin by recalling the Berberian representation. Following [1], we denote by Lim a fixed Banach generalized limit, defined for bounded sequences $\{\lambda_n\}$ of complex numbers; thus

$$\operatorname{Lim}(\lambda_n + \mu_n) = \operatorname{Lim}\lambda_n + \operatorname{Lim}\mu_n, \tag{5.1}$$

$$\operatorname{Lim}(\lambda\lambda_n) = \lambda \operatorname{Lim}\lambda_n, \tag{5.2}$$

$$\operatorname{Lim}\lambda_n = \lim\lambda_n \quad \text{whenever } \{\lambda_n\} \text{ is convergent}, \tag{5.3}$$

$$\operatorname{Lim}\lambda_n \geq 0 \quad \text{when } \lambda_n \geq 0 \text{ for all } n. \tag{5.4}$$

We note that the translation invariant property of Lim is not assumed here.

Denote by V the set of all bounded sequences $x = \{x_n\}$ with x_n in H. It is clear that V is a vector space relative to the definitions $x+y = \{x_n+y_n\}$ and $\lambda x = \{\lambda x_n\}$ for $x = \{x_n\}$ and $y = \{y_n\}$. Let N be the set of all null sequences $\{x_n\}$ such as $\mathrm{Lim}(x_n, y_n) = 0$ for all $\{y_n\} \in V$. Clearly N is a linear subspace of V; we write x' for the coset $x + N$. The quotient vector space V/N becomes an inner product space on defining $(x', y') = \mathrm{Lim}(x_n, y_n)$. If $x \in H$, then $\{x\}$ means the sequence of all whose terms are x. Since $(x', y') = (x, y)$ for $x' = \{x\}+N$ and $y' = \{y\}+N$, the mapping $x \to x'$ is an isometric linear map of H onto the closed subspace H' of V/N. Let H^0 be the completion of V/N. Then H^0 is an extension of H. For an operator A acting on H, put

$$A^0(\{x_n\} + N) = \{Ax_n\} + N.$$

We can extend A^0 on H^0, which will be denoted by A^0 too. The mapping $A \to A^0$ of $B(H)$ into $B(H^0)$ is called the Berberian representation. The following theorem is proved in [1].

Theorem B (Berberian). *The Berberian representation is *-isomorphic and isometric. If $A \in B(H)$, then $\pi(A) = \pi(A^0) = \sigma_p(A^0)$, where $\pi(A)$ is the approximate point spectrum of A and $\sigma_p(A)$ is the point spectrum of A.*

As a matter of fact, Theorem B is used as the following form stated in the proof of [4, Theorem 1].

Theorem C. *The Berberian representation converts every normal approximate eigenvalue of A into the normal eigenvalue of A^0.*

Proof of Theorem 3. For each n, let f_n be orthogonal vectors to e_n, and let α_n be complex numbers such that $g = \alpha_n e_n + f_n$. Here, let m be a real vector space of bounded sequences of real numbers. Put $p(x) = \overline{\lim}\, x_n$ for $x = (x_n) \in m$. By the Hahn-Banach theorem, for any $x_0 \in m$ there is a positive linear form F on m such that $p(x_0) = F(x_0)$ and $-p(-x) \le F(x) \le p(x)$ for all $x \in m$, and F is clearly extended on a complex vector space of bounded sequences of complex numbers. Therefore, for a bounded sequence $\{|\alpha_n|^2\}$ there is a Banach generalized limit Lim such that $\mathrm{Lim}\,|\alpha_n|^2 = \overline{\lim}\,|\alpha_n|^2$. Put $g' = \{g\}$, $e = \{e_n\}$, $f = \{f_n\}$ in H^0, and $\alpha = \mathrm{Lim}\,\alpha_n$; then e is a unit vector orthogonal to f and $g' = \alpha e + f$. Also we have

$$|\alpha|^2 + \|f\|^2 = \|g'\|^2 = \mathrm{Lim}(|\alpha_n|^2 + \|f_n\|^2) = \mathrm{Lim}\,|\alpha_n|^2 + \|f\|^2,$$

that is, $|(g', e)|^2 = |\alpha|^2 = \mathrm{Lim}\,|\alpha_n|^2$. Furthermore, it follows from Theorem C that λ is a normal eigenvalue of A^0, i.e., $A^0 e = \lambda e$ and $A^{0*} e = \lambda^* e$. Finally Theorem 1 implies that

$$\begin{aligned}\frac{\|g\|^2\|Ag\|^2 - |(g, Ag)|^2}{\|(A-\lambda)g\|^2} &= \frac{\|g'\|^2\|A^0 g'\|^2 - |(g', A^0 g')|^2}{\|(A^0-\lambda)g'\|^2} \\ &\ge |(g', e)|^2 = \mathrm{Lim}\,|(g, e_n)|^2 = \overline{\lim}\,|(g, e_n)|^2.\end{aligned}$$

Remark. (1) As in the proof above, put $B = A - \lambda$. Then $\|B^0 f\| = \|B^0 g'\| = \|Bg\| \ne 0$, so $f \ne 0$. Therefore the equation holds in (4) if and only if f is also an eigenvector of B^0, that is, $\{f_n\}$ is a sequence of approximate eigenvectors of A.

(2) Finally we show a simple example of a nondominant operator, which can be covered by Theorem 3. That is, Theorem 3 is applicable to a wider class of operators than the one of dominant operators. Actually let M be the multiplication operator on $L^2[0, 1]$, i.e., $(Mf)(t) = tf(t)$ for $f \in L^2[0, 1]$ and

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus M.$$

Then the set of all normal approximate eigenvalues of T is the interval $[0, 1]$ and T is not dominant because the matrix is not dominant.

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