AN INEQUALITY FOR SOME NONNORMAL OPERATORS—EXTENSION TO NORMAL APPROXIMATE EIGENVALUES

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ABSTRACT. An inequality of use in testing convergence of eigenvector calculations is extended. It is an improvement of Bernstein and Furuta’s results for selfadjoint operators and dominant operators respectively.

1. Introduction

In [5] one of the authors showed the following theorem, which is an improvement of Bernstein’s in [2].

Theorem A. If \( e \) is a unit eigenvector corresponding to an eigenvalue \( \lambda \) in a dominant operator \( A \) on a Hilbert space \( H \), then

\[
\langle g, e \rangle \leq \frac{\|g\|^2 \|Ag\|^2 - \langle g, Ag \rangle^2}{\|(A - \lambda)g\|^2}
\]

for all \( g \) in \( H \) for which \( Ag \neq \lambda g \).

Here an operator \( A \) is called dominant if for each \( \lambda \) there is a real number \( M_\lambda > 1 \) such that \( \|(A - \lambda)^*x\| \leq M_\lambda \|(A - \lambda)x\| \) for all \( x \) in \( H \). We have to remark that \( (A - \lambda)^*e = 0 \) under the dominance of \( A \). This says that \( \lambda \) is a normal eigenvalue of \( A \), i.e., there is a nonzero vector \( x \) in \( H \) such that

\[
(A - \lambda)x = 0 \quad \text{and} \quad (A - \lambda)^*x = 0.
\]

In this note, we weaken the assumption of Theorem A to dominance of the operator to normality of the eigenvalue. More precisely,

Theorem 1. If \( e \) is a unit eigenvector corresponding to a normal eigenvalue \( \lambda \) of \( A \) on a Hilbert space \( H \), then (1) holds for all \( g \) in \( H \) for which \( Ag \neq \lambda g \).

Second, we give another generalization of Theorem A to normal approximate eigenvalues [3], i.e., a complex number \( \lambda \) is called a normal approximate eigenvalue of \( A \) if there exists a sequence \( \{x_n\} \) of unit vectors such that

\[
\| (A - \lambda)x_n \| \to 0 \quad \text{and} \quad \| (A - \lambda)^*x_n \| \to 0.
\]

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It is known in [4] that the correspondence between the normal approximate eigenvalues and the normal eigenvalues is ensured by the Berberian representations in [1]. Therefore there is a bridge from Theorem 1 to this generalization, which is of course supported by the Berberian one.

2. Results

First of all, we cite the following lemma, which is one of the technical points in [2, 5].

**Lemma 2.** Let \( A \) be an operator on a Hilbert space \( H \). Then

\[
\|x\|^2 \|Ax\|^2 - |(x, Ax)|^2 = \|x\|^2 \|(A - \lambda)x\|^2 - |(x, (A - \lambda)x)|^2
\]

for all \( x \) in \( H \) and complex numbers \( \lambda \).

**Proof of Theorem 1.** Though the proof is almost the same as in [5], we give a proof for the sake of completeness. A given \( g \) in \( H \) is expressed as \( g = \alpha e + f \), where \( \alpha = (g, e) \) and \( (e, f) = 0 \). Put \( B = A - \lambda \). Then \( f \neq 0 \), \( Bg = Bf \), so

\[
(g,Bg) = (g,Bf) = \alpha (e,Bf) + (f,Bf) = (f,Bf)
\]

because \( B^*e = 0 \). Hence it follows from Lemma 1 that

\[
\frac{\|g\|^2 \|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2} = \frac{(|\alpha|^2 + \|f\|^2)\|Bg\|^2 - |(g, Bg)|^2}{\|Bg\|^2} = |\alpha|^2 + \frac{\|f\|^2 \|Bf\|^2 - |(f, Bf)|^2}{\|Bf\|^2} \geq |\alpha|^2.
\]

**Remark.** As in the above proof, the equality holds in (1) if and only if \( f \) is also an eigenvector of \( A \).

Next we consider a generalization of Theorem 1 to normal approximate eigenvalues. Here we call the \( \{e_n\} \) satisfying (3) a sequence of unit eigenvectors corresponding to a normal approximate eigenvalue \( \lambda \) of \( A \).

**Theorem 3.** If \( \{e_n\} \) is a sequence of unit vectors corresponding to a normal approximate eigenvalue \( \lambda \) of \( A \), then

\[
\lim |(g, e_n)|^2 \leq \frac{\|g\|^2 \|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2}
\]

for all \( g \) in \( H \) for which \( Ag \neq \lambda g \).

A proof of Theorem 3 will be presented in the next section.

3. The Berberian representation

We begin by recalling the Berberian representation. Following [1], we denote by \( \operatorname{Lim} \) a fixed Banach generalized limit, defined for bounded sequences \( \{\lambda_n\} \) of complex numbers; thus

\begin{align}
(5.1) \quad & \operatorname{Lim}(\lambda_n + \mu_n) = \operatorname{Lim}\lambda_n + \operatorname{Lim}\mu_n, \\
(5.2) \quad & \operatorname{Lim}(\lambda\lambda_n) = \lambda \operatorname{Lim}\lambda_n, \\
(5.3) \quad & \operatorname{Lim}\lambda_n = \lim\lambda_n \quad \text{whenever} \ \{\lambda_n\} \ \text{is convergent,} \\
(5.4) \quad & \operatorname{Lim}\lambda_n \geq 0 \quad \text{when} \ \lambda_n \geq 0 \ \text{for all} \ n.
\end{align}
We note that the translation invariant property of Lim is not assumed here.

Denote by \( V \) the set of all bounded sequences \( x = \{x_n\} \) with \( x_n \) in \( H \). It is clear that \( V \) is a vector space relative to the definitions \( x + y = \{x_n + y_n\} \) and \( \lambda x = \{\lambda x_n\} \) for \( x = \{x_n\} \) and \( y = \{y_n\} \). Let \( N \) be the set of all null sequences \( \{x_n\} \) such as \( \text{Lim}(x_n, y_n) = 0 \) for all \( \{y_n\} \in V \). Clearly \( N \) is a linear subspace of \( V \); we write \( x' \) for the coset \( x + N \). The quotient vector space \( V/N \) becomes an inner product space on defining \( (x', y') = \text{Lim}(x_n, y_n) \). If \( x \in H \), then \( \{x\} \) means the sequence of all whose terms are \( x \). Since \( (x', y') = (x, y) \) for \( x' = \{x\} + N \) and \( y' = \{y\} + N \), the mapping \( x \mapsto x' \) is an isometric linear map of \( H \) onto the closed subspace \( H' \) of \( V/N \). Let \( H^0 \) be the completion of \( V/N \). Then \( H^0 \) is an extension of \( H \). For an operator \( A \) acting on \( H \), put

\[
A^0(\{x_n\} + N) = \{Ax_n\} + N.
\]

We can extend \( A^0 \) on \( H^0 \), which will be denoted by \( A^0 \) too. The mapping \( A \to A^0 \) of \( B(H) \) into \( B(H^0) \) is called the Berberian representation. The following theorem is proved in [1].

**Theorem B (Berberian).** The Berberian representation is \(^*\)-isomorphic and isometric. If \( A \in B(H) \), then \( \pi(A) = \pi(A^0) = \sigma_p(A^0) \), where \( \pi(A) \) is the approximate point spectrum of \( A \) and \( \sigma_p(A) \) is the point spectrum of \( A \).

As a matter of fact, Theorem B is used as the following form stated in the proof of [4, Theorem 1].

**Theorem C.** The Berberian representation converts every normal approximate eigenvalue of \( A \) into the normal eigenvalue of \( A^0 \).

**Proof of Theorem 3.** For each \( n \), let \( f_n \) be orthogonal vectors to \( e_n \), and let \( \alpha_n \) be complex numbers such that \( g = \alpha_n e_n + f_n \). Here, let \( m \) be a real vector space of bounded sequences of real numbers. Put \( p(x) = \lim x_n \) for \( x = \{x_n\} \in m \).

By the Hahn-Banach theorem, for any \( x_0 \in m \) there is a positive linear form \( F \) on \( m \) such that \( p(x_0) = F(x_0) \) and \( -p(-x) \leq F(x) \leq p(x) \) for all \( x \in m \), and \( F \) is clearly extended on a complex vector space of bounded sequences of complex numbers. Therefore, for a bounded sequence \( \{\alpha_n^2\} \) there is a Banach generalized limit \( \lim \) such that \( \lim |\alpha_n|^2 = \lim |\alpha_n|^2 \). Put \( g' = \{g\} \), \( e = \{e_n\} \), \( f = \{f_n\} \) in \( H^0 \), and \( \alpha = \lim \alpha_n \); then \( e \) is a unit vector orthogonal to \( f \) and \( g' = \alpha e + f \). Also we have

\[
|\alpha|^2 + \|f\|^2 = \|g'\|^2 = \lim(|\alpha_n|^2 + \|f_n\|^2) = \lim |\alpha_n|^2 + \|f\|^2,
\]

that is, \( |(g', e)|^2 = |\alpha|^2 = \lim |\alpha_n|^2 \). Furthermore, it follows from Theorem C that \( \lambda \) is a normal eigenvalue of \( A^0 \), i.e., \( A^0 e = \lambda e \) and \( A^0 e = \lambda^* e \). Finally Theorem 1 implies that

\[
\frac{\|g\|^2 \|Ag\|^2 - |(g, Ag)|^2}{\|(A - \lambda)g\|^2} = \frac{\|g'\|^2 \|A^0 g'\|^2 - |(g', A^0 g')|^2}{\|(A^0 - \lambda)g'\|^2} \geq |(g', e)|^2 = \lim |(g, e_n)|^2 = \lim |(g, e_n)|^2.
\]

**Remark.** (1) As in the proof above, put \( B = A - \lambda \). Then \( \|B^0 f\| = \|B^0 g'\| = \|Bg\| \neq 0 \), so \( f \neq 0 \). Therefore the equation holds in (4) if and only if \( f \) is also an eigenvector of \( B^0 \), that is, \( \{f_n\} \) is a sequence of approximate eigenvectors of \( A \).
Finally we show a simple example of a nondominant operator, which can be covered by Theorem 3. That is, Theorem 3 is applicable to a wider class of operators than the one of dominant operators. Actually let $M$ be the multiplication operator on $L^2[0, 1]$, i.e., $(Mf)(t) = tf(t)$ for $f \in L^2[0, 1]$ and
\[ T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus M. \]
Then the set of all normal approximate eigenvalues of $T$ is the interval $[0, 1]$ and $T$ is not dominant because the matrix is not dominant.

References