FIXED POINTS OF FINITE GROUPS OF FREE GROUP AUTOMORPHISMS

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Abstract. We construct an equivariant collection of contractions of the compactified Culler-Vogtmann outer space \( \overline{X}_n \). As a consequence, we prove that any finite subgroup of the outer automorphism group of a free group fixes a contractible subset of \( \overline{X}_n \).

1. Introduction

Culler and Vogtmann [CV] initiated a study of the outer automorphism group \( \text{Out}(F_n) \) of the free group \( F_n \) on \( n \) letters by constructing a space \( X_n \) upon which \( \text{Out}(F_n) \) acts properly discontinuously. This space, which has since come to be known as the “outer space”, consists of projective classes of free minimal actions of \( F_n \) on simplicial metric trees, where two such trees are identified if they are equivariantly isometric. The quotient of each such tree by such an action is a finite marked metric graph in which each vertex has valence \( \geq 3 \). Here, “metric” means that each edge has a length, and a “marking” is a preferred homotopy equivalence of the wedge \( R_n \) of \( n \) circles to this graph. One thus has two possible views of \( X_n \); these are interchangeable. The group \( \text{Out}(F_n) \) acts in the obvious way: one represents an automorphism of \( F_n \) as a self-map of \( R_n \) and precomposes the marking with this map.

Culler and Vogtmann showed that \( X_n \) is a finite-dimensional contractible space and that the \( \text{Out}(F_n) \)-action has finite stabilizers and finite quotient. Furthermore, \( X_n \) has a natural compactification \( \overline{X}_n \) in which the boundary consists of certain actions of \( F_n \) on \( \mathbb{R} \)-trees with cyclic edge stabilizers. It is known that \( \overline{X}_n \) is contractible; this was first shown by Steiner [St] and, independently, by Skora [Sk]. Skora's technique was to construct a path between two \( \mathbb{R} \)-trees, given a morphism between them. (A morphism is a map such that each segment in the domain \( \mathbb{R} \)-tree contains an initial arc which is mapped isometrically.) In this paper we generalize Skora’s methods to prove the following theorem:

**Theorem.** Let \( \mathcal{X} \) be the space of nontrivial semisimple actions of \( F_n \) on \( \mathbb{R} \)-trees. Then there is a continuous deformation \( F : X_n \times \mathcal{X} \times I \rightarrow \mathcal{X} \) such that

\[ F(T_0, T_1, 0) = T_0; \]

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(2) \( F(T_0, T_1, t) = T_1 \); \\
(3) \( F(T_0, T_0, t) = T_0 \) for all \( t \); and \\
(4) \( F \) is equivariant under the diagonal action of \( \text{Out}(F_n) \) on \( X_n \times X' \).

The result also holds if we replace \( X' \) by the compactification \( \overline{X}_n \subset X' \) of \( X_n \). An immediate consequence is the following:

**Corollary.** The subset \( X_G \) (resp. \( \overline{X}_G \)) of \( X_n \) (resp. \( \overline{X}_n \)) fixed by a finite subgroup \( G \) of \( \text{Out}(F_n) \) is contractible.

### 2. Preliminaries

**Group actions on R-trees.** An R-tree is a metric space such that any two points are joined by a unique embedded arc, which is isometric to a finite interval in the real line. For a general exposition of the theory of group actions on R-trees, see, for example, [CM].

Let \( G \times T \to T \) be an action of a finitely generated group \( G \) on an R-tree. The metric on \( T \) will be denoted by \( d \) or \( d_T \) as the context requires. The **translation length function** \( l(g) \) determined by the action is the map \( l : G \to \mathbb{R} \) defined by \( l(g) = \min_{x \in T} d(x, gx) \). The **characteristic set** of \( g \) is \( \mathcal{T}_g = \{ x \in T \mid d(x, gx) = l(g) \} \). Since \( l(g) = l(xgx^{-1}) \) for all \( g, x \in G \), we may consider \( l \) as a real-valued function on the set \( C \) of conjugacy classes in \( G \). Actions of \( G \) on R-trees \( T \) and \( T' \) are **projectively equivalent** if there is an equivariant homeomorphism from \( T \) to \( T' \) such that the induced pullback metric on \( T \) differs from the original metric by a nonzero multiplicative constant.

An action is **irreducible** if no finite set of ends of \( T \) is invariant under \( G \) and **dihedral** if \( G \) interchanges a pair of ends. The action is **semisimple** if it is irreducible, dihedral, or trivial.

We now describe a topology, due to Paulin [P], on the space of actions of \( G \) on R-trees. An **\( \varepsilon \)-approximation** between two compact metric spaces \( K \) and \( L \) is a subdirect product \( R \subset K \times L \) (i.e., the relation \( R \) surjects onto both \( K \) and \( L \)) such that if \( x_0Ry_0 \) and \( x_1Ry_1 \), then \( |d(x_0, x_1) - d(y_0, y_1)| < \varepsilon \). Following Gromov [G], define \( d_G(K, L) < \varepsilon \) if there exists an \( \varepsilon \)-approximation between \( K \) and \( L \). More generally, suppose \( X \) is a G-metric space (i.e., a metric space with an associated G-action) and that \( P \) is a finite subset of \( G \). An **\( \varepsilon \)-approximation** is \( P \)-**equivariant** if whenever \( g \in P, x \in K, gx \in K, y \in L, \) and \( xRy \), then \( gy \in L \) and \( gxRgy \).

Now fix a compact \( K \subset X \), a finite subset \( P \) of \( G \), and \( \varepsilon > 0 \). Paulin [P] defines the basic neighborhood \( U(X, K, P, \varepsilon) \) to be the collection of G-metric spaces \( Y \) such that there exists a compact \( L \subset Y \) and a \( P \)-equivariant \( \varepsilon \)-approximation from \( K \) to \( L \) and uses these as a neighborhood base for a topology on G-metric spaces.

Skora [Sk] generalizes this to define a topology on a space \( \mathcal{B} \) of equivariant maps between G-metric spaces. We give the definitions here for completeness. If \( \phi : X \to X' \) and \( \psi : Y \to Y' \) are maps, an **\( \varepsilon \)-approximation** from \( \phi \) to \( \psi \) is a pair of relations \( (R, R') \) such that

1. \( R \) and \( R' \) are \( \varepsilon \)-approximations from \( X \) to \( Y \) and from \( X' \) to \( Y' \) respectively; and
2. \( xRy \) implies \( \phi(x)RR'\psi(y) \).
We endow a product $X \times X'$ with the uniform metric:

$$d((x_0, x'_0), (x_1, x'_1)) = \max\{d(x_0, x_1), d(x'_0, x'_1)\}.$$ 

Given $\epsilon$-approximations $R$ from $X$ to $Y$ and $R'$ from $X'$ to $Y'$, one sees that $(R, R')$ is an $\epsilon$-approximation from $X \times X'$ to $Y \times Y'$, where $(R, R')$ is the product of $R$ and $R'$ as subsets of $X \times Y$ and $X' \times Y'$ respectively.

The notion of $P$-equivariance extends naturally to the case of $\epsilon$-approximations between maps. Skora defines a topology on a space $C$ of equivariant maps between $G$-metric spaces as follows: Let $K \times K'$ be a compact subset of $X \times X'$, $P$ a finite subset of $G$, and $\epsilon > 0$. The basic neighborhood $U(\phi, K \times K', P, \epsilon)$ of $\phi$ is the set of all maps $\psi : Y \to Y'$ in $C$ such that for some compact $L \times L' \subseteq Y \times Y'$, there is a $P$-equivariant, closed $\epsilon$-approximation from $\phi|K$ to $\psi|L$. $C$ is then given the topology generated by these basic open sets.

A universal bundle. Fix a group $G$ and a space $\mathcal{X}$ of actions of $G$ on $\mathbb{R}$-trees. In this section we introduce a bundle over $\mathcal{X}$ which should be thought of as an analogue of the universal Teichmüller bundle over Teichmüller space. Although it is essentially a notational convenience, this bundle will be useful in describing some constructions later and in dealing with the assorted group actions which arise.

**Definition 2.1.** Let $\mathcal{X}$ be a space of $G$-actions on $\mathbb{R}$-trees. The tree bundle $\mathcal{F}(\mathcal{X})$ over $\mathcal{X}$ is the space $\{(T, x) \mid T \in \mathcal{X} \text{ and } x \in T\}$. If the $\mathbb{R}$-trees are simplicial, we further define the vertex bundle $\mathcal{V}(\mathcal{X})$ over $\mathcal{X}$ to be the space $\{(T, v) \mid T \in \mathcal{X} \text{ and } v \text{ is a vertex of } T\}$. There is a natural $G$-action on each fiber of these bundles.

$\text{Aut}(G)$ acts naturally on $\mathcal{F}(\mathcal{X})$ by permuting the fibers. If we fix a group action $G \times T \to T$ in $\mathcal{X}$ and $\phi \in \text{Aut}(G)$, we can define the image of this action under $\phi$ by precomposing:

$$G \times T \xrightarrow{\gamma} T \xmapsto{\phi \circ (x \times \text{id})} T.$$ 

In $\mathcal{F}(\mathcal{X})$, then, we define $\phi(T, x) = (T', x')$ if there exists an isometry of pointed spaces $f : (T, x) \to (T', x')$ which induces $\phi$ in the sense that the following diagram commutes:

$$\begin{array}{ccc}
G \times T & \xrightarrow{\phi \times f} & G \times T' \\
\downarrow & & \downarrow \\
T & \xrightarrow{f} & T'
\end{array}$$

This action restricts to an action on $\mathcal{V}(\mathcal{X})$. Note that the "vertical" action of $G$ on the fibers is simply the restriction of the $\text{Aut}(G)$ action to $\text{Inn}(G)$, and since $\text{Inn}(G)$ acts trivially on the base space, we recover the standard $\text{Out}(G)$ action on $\mathcal{X}$.

Finally, there is a natural topology on the tree bundle. $T$ and $T'$ are close if there exists an approximating relation $R$ between large compact subsets of $T$ and $T'$, such that $R$ is equivariant with respect to a large finite subset of $G$. Hence two points $(T, x)$ and $(T', x')$ in the
bundle are close if $xRx'$ for such a relation $R$. We endow the vertex bundle with the topology it inherits from the tree bundle.

3. Weighted length functions

Throughout this section, $G$ denotes a finitely generated group. All actions of $G$ are assumed to be minimal.

We now generalize the notions of translation length function and characteristic set. Let $G$ act on $T$, and let $\lambda$ be a real-valued function on $G$. The weighted length function $f_{\lambda}: T \to \mathbb{R}$ is defined to be $\sup_{g \in G} \lambda(g)d(x, gx)$. We define the length of $\lambda$ to be the infimum of $f_{\lambda}$: $l(\lambda) = l_{T}(\lambda) = \inf_{x \in T} f_{\lambda}(x)$. (We will omit the subscript if the tree is clear from the context.) The characteristic set of $\lambda$ is $T_{\lambda} = \{ x \in T | f_{\lambda}(x) = l(\lambda) \}$.

Remarks. 1. For a general function $\lambda$, it may happen that $l(\lambda) = \infty$ and that $T_{\lambda} = T$. We will eventually restrict our attention to functions $\lambda$ for which this does not occur, but remark that the definitions still make sense in this case.

2. Skora [Sk] considers the case in which $\lambda$ is the characteristic function of a finite generating set of $G$. The arguments here have a slightly different flavor, since the weighting functions now have infinite support.

3. The standard translation length $l(g)$ of a group element $g$ is recovered in this context by taking $\lambda$ to be the function taking the value 1 at $g$ and 0 elsewhere. Just as $l(g)$ is invariant under conjugacy in $G$, the length $l(\lambda)$ is invariant under the “inner” action of $G$ on $\mathbb{R}^G$ by conjugation.

There is a natural action of $\text{Aut}(G)$ on $\mathbb{R}^G$: given $\lambda \in \mathbb{R}^G$ and an automorphism $\phi$ of $G$, one can define $\phi(\lambda) = \lambda \circ \phi^{-1}$. The following lemma is immediate.

Lemma 3.1. If $\phi \in \text{Aut}(G)$, $\lambda \in \mathbb{R}^G$, and $f$ is an isometry from $(T, v)$ to $(T', v')$ inducing $\phi$ as in the preceding section, then:

1. $l_{f(T)}(\lambda) = l_{T}(\phi(\lambda))$, and
2. $T_{\phi(\lambda)} = f(T_{\lambda})$.

In other words, the generalized notions of length function and characteristic set are equivariant. □

For the remainder of this section, we shall assume $G$ acts freely on a simplicial $\mathbb{R}$-tree $T$ (and hence $G = F_n$). We now choose, for each point of $T$, a weighting function on $G$:

Definition. Let $G \times T \to T$ be a free simplicial action. For any $x \in T$, define a weighting function $\lambda_{T, x}: G \to \mathbb{R}$ by $\lambda_{T, x}(g) = 1/d_T(x, gx)$ if $g \neq \text{id}$, and $\lambda_{T, x}(\text{id}) = 0$.

Then, for any semisimple action $G \times T' \to T'$, we let $f_{T, x}: T \to \mathbb{R}$ denote the weighted length function corresponding to $\lambda_{T, x}$:

$$f_{T, x}(y) = \sup_{g \in G} \lambda_{T, x}(g)d(y, gx) = \sup_{g \in G \setminus \{\text{id}\}} \frac{d_T(y, gx)}{d_T(x, gx)}.$$

We first analyze the behavior of the function $f_{T, x}$ on $T$. A priori, the supremum defining $f_{T, x}$ may not even be finite, so this is the first issue to be dealt with.
Lemma 3.2. Suppose that $G$ acts irreducibly on $T$. Then for each $x \in T$, the function $f_{T,x} : T \to \mathbb{R}$ is:

1. finite,
2. proper; and
3. has a unique minimum at $x$.

Proof. (1) Fix $y \in T$. Then for any $g \in G$,

$$d(y, gy) \leq d(x, y) + d(x, gx) + d(gx, gy) = d(x, gx) + 2d(x, y),$$

so that $\lambda_{T,x}(g)d(y, gy) \leq 1 + 2d(x, y)/d(x, gx)$. But the action on $T$ is free and simplicial, so $d(x, gx)$ is uniformly bounded away from zero for all $g \neq id$. Thus $\lambda_{T,x}(g)d(y, gy)$ is uniformly bounded in $g$, and thus $f_{T,x}(y) < \infty$.

(2) Let $y$ be any point in $T$ other than $x$. By the minimality of $T$, we can choose an element $g$ of $G$ such that $x$ lies on the segment joining $y$ to the axis of $G$. For such a $g$, $d(y, gy) = d(x, gx) + 2d(x, y)$. Hence $f_{T,x}(y)$ increases with $d(x, y)$. This also proves (3). □

We shall require the following lemma:

Lemma 3.3. Let $\lambda : G \to \mathbb{R}$ be a real-valued function. Suppose that $G \times T \to T$ is a free simplicial action such that the weighted length function $f_\lambda$ is finite and proper on $T$. Then $f_\lambda$ is finite and proper for any semisimple action of $G$ on an $\mathbb{R}$-tree $T'$.

Proof. Let $h : T \to T'$ be an edgewise-linear equivariant map. We use $h$ to relate the behavior of $f_\lambda$ on $T'$ to its behavior on $T$. Since $T$ has compact quotient, $h$ is $L$-Lipschitz for some $L > 0$. Furthermore, $h$ is onto, since the action on $T'$ is minimal. Let $z \in T$ be an $h$-preimage of $y \in T'$. Then $d_{T'}(y, gy) \leq Ld_T(z, gz)$ for any $g \in G$. Thus $f_\lambda(y) \leq Lf_\lambda(z)$. By the preceding lemma, $f_\lambda$ is finite on $T$, so it is finite on $T'$. That $f_\lambda$ is proper follows as in the preceding lemma. □

Definition 3.4. Let $\Lambda$ denote the set of real-valued functions $\lambda$ on $G$ such that the weighted length function $f_\lambda(x)$ is a proper, real-valued function for a free action of $G$ on a simplicial $\mathbb{R}$-tree $T$. Since any two such actions are equivariantly quasi-isometric, the preceding lemma implies that $\Lambda$ is independent of the choice of $T$. We remark that the weighted length function is automatically convex for each $\lambda \in \Lambda$, as each function $d_T(x, gx)$ is convex.

Remark. The correspondence $(T, x) \mapsto \lambda_{T,x}$ gives a continuous map from the tree bundle $\mathcal{F}(X_n)$ to $\mathbb{R}^G$. Lemma 3.2 thus says that the image of this map is contained in $\Lambda$.

Now suppose $\lambda \in \Lambda$, and let $T$ be an $\mathbb{R}$-tree equipped with a semi-simple action of $G$. Since $f_\lambda$ is convex and proper on $T$, its characteristic set $T_\lambda$ is a nonempty compact subtree of $T$. We can then choose a "base point" $b(T, \lambda)$ in $T$ by selecting the "center" of $T_\lambda$; that is, the unique point $y$ such that the ball at $y$ of radius $(\text{diam} T_\lambda)/2$ is contained in $T_\lambda$. Hence $b(\cdot, \lambda)$ may be thought of as a section of the tree bundle $\mathcal{F}(\mathcal{X})$. The following shows that these sections are continuous and vary continuously with $\lambda$.

Proposition 3.5. Let $G = F_n$ and $\Lambda$ be as above. Let $\mathcal{X}$ denote the space of semisimple actions of $G$ on $\mathbb{R}$-trees and $\mathcal{F}(\mathcal{X})$ be the tree bundle over $\mathcal{X}$. Then the associated base point function $b : \mathcal{X} \times \Lambda \to \mathcal{F}(\mathcal{X})$ is continuous.
The proof is essentially identical to that of Proposition 5.2 in [Sk]. More generally, if \( G \) is a finitely generated group, \( \mathcal{H} \) a space of actions of \( G \) on \( \mathbb{R} \)-trees, and \( \Lambda \) a collection of weight functions on \( G \) such that \( f_\lambda \) is finite and proper for each \( \lambda \in \Lambda \) and each action in \( \mathcal{H} \), then the same result holds.

**Lemma 3.6.** \( b \) is equivariant under the action of \( \text{Aut}(F_n) \) on \( \mathcal{H} \times \Lambda \).

**Proof.** This follows immediately from Lemma 3.1. \( \square \)

### 4. Paths between \( \mathbb{R} \)-trees

In this section we take \( G \) to be the free group \( F_n \) of a fixed rank \( n \). Recall [CV] that “outer space” is the space \( X_n \) of projective classes of minimal free actions of \( F_n \) on simplicial \( \mathbb{R} \)-trees. A map from a simplicial \( \mathbb{R} \)-tree to an arbitrary \( \mathbb{R} \)-tree is *transverse* if it is linear on each 1-simplex.

If \( f \) is any function, we will denote by \( \mathcal{D}(f) \) and \( \mathcal{R}(f) \) the domain and range of \( f \).

**Proposition 4.1.** Let \( \mathcal{H} \) be a space of nontrivial irreducible actions of \( F_n \) on \( \mathbb{R} \)-trees, and let \( \mathcal{C} = \mathcal{C}(X_n, \mathcal{H}) \) be the space of all equivariant transverse maps from simplicial \( \mathbb{R} \)-trees to elements of \( \mathcal{H} \). Then there is a map \( B: X_n \times \mathcal{H} \rightarrow \mathcal{C} \) such that \( \mathcal{D}(B(Y, T)) = Y \), \( \mathcal{R}(B(Y, T)) = T \), and \( B(Y, Y) = \text{Id}_Y \) for all \( Y \in X_n \cap \mathcal{H} \), \( T \in \mathcal{H} \). Furthermore, \( B \) is equivariant with respect to the diagonal action of \( \text{Out}(F_n) \) on \( X_n \times \mathcal{H} \).

**Proof.** Given \( Y \in X_n \) and \( T \in \mathcal{H} \), we use the base point map to define the image of a vertex \( v \) of \( Y \): \( B(Y, T)(v) = b(T, \lambda_Y, v) \). Now extend the domain of \( B(Y, T) \) to all of \( Y \) by mapping linearly on the edges of \( Y \). Now \( \lambda_Y, v \) is continuous, so \( B(Y, T) \) is continuous by Lemma 3.3 and Proposition 3.5. \( B(Y, T) \) is equivariant under \( \text{Aut}(F_n) \) by Lemma 3.6. Finally, by Lemma 3.2, the weighted length function \( f_\lambda, v \) on \( Y \) has a unique minimum at \( v \), so \( B(Y, Y) \) is the identity on the vertices of \( Y \) and hence on all of \( Y \) by linearity. \( \square \)

**Remark.** In the above proposition, it is essential that the action on the source \( \mathbb{R} \)-tree be free. If any \( g \in F_n \) has a fixed point in \( Y \), it is impossible in general to construct an equivariant map from \( Y \) to \( T \).

Recall [Sk] that an (equivariant) *morphism* between \( \mathbb{R} \)-trees is an (equivariant) map such that each segment in the source \( \mathbb{R} \)-tree has a nontrivial initial segment which is mapped isometrically onto its image. Given a surjective morphism \( \phi: T_0 \rightarrow T_1 \), Skora constructs morphisms \( \phi_{st} \) for \( 0 \leq s \leq t \leq 1 \) such that \( \phi_{01} = \phi \), \( \phi_{st} \circ \phi_{rs} = \phi_{rt} \), and \( \phi_{rs} = \text{id} \) whenever \( \phi = \text{id} \). We will need the following result:

**Proposition 4.2 [Sk, 4.8].** Let \( G \) be a group, \( \mathcal{H} \) the space of all actions of \( G \) on \( \mathbb{R} \)-trees, and \( \mathcal{C}(\mathcal{H}) \) the space of all morphisms between elements of \( \mathcal{H} \). Then the function

\[
\mathcal{C}(\mathcal{H}) \times \{(s, t) \mid 0 \leq s \leq t \leq 1\} \rightarrow \mathcal{C}(\mathcal{H})
\]

defined via \((\phi, (s, t)) \mapsto \phi_{st}\) is continuous. Furthermore, this map is equivariant with respect to the natural action of \( \text{Out}(F_n) \) on \( \mathcal{C}(\mathcal{H}) \).

Hence a morphism between two \( \mathbb{R} \)-trees determines a canonical path between them. We now improve Proposition 4.1 to promote \( B(Y, T) \) to a morphism:
Proposition 4.3. Let \( \mathcal{H} \) be the space of nontrivial semisimple actions of \( F_n \) on \( \mathbb{R} \)-trees, and let \( \mathcal{C} = \mathcal{C}(\overline{X}_n, \mathcal{H}) \) be the space of all equivariant transverse morphisms from elements of \( \overline{X}_n \) to elements of \( \mathcal{H} \). Then there is a map \( M : \overline{X}_n \times \mathcal{H} \to \mathcal{C} \) such that the following hold for all \( Y \in \overline{X}_n, T \in \mathcal{H} \):

1. \( \mathcal{H}(M(Y, T)) = T \);
2. \( \mathcal{H}(M(Y, T)) \) is a simplicial \( \mathbb{R} \)-tree contained in the closure of the simplex of \( \overline{X}_n \) containing \( Y \);
3. if \( T \) is a free simplicial action, so is \( \mathcal{H}(M(Y, T)) \);
4. \( M(Y, Y) = \text{Id}_Y \); and
5. \( M \) is equivariant under the diagonal action of Out\( (F_n) \).

Proof. For each 1-simplex \( \sigma \) in \( Y \), \( B(Y, T) \) maps \( \sigma \) linearly into \( T \), dilating by some nonnegative number \( \mu \). Multiply the metric on \( \sigma \) by \( \mu \). Let \( Y' \) be the tree obtained from \( Y \) in this way. \( B(Y, T) \) then induces a morphism \( M(Y, T) \) from \( Y' \) to \( T \). \( M \) clearly varies continuously with \( Y \) and \( T \), and the conclusions all follow immediately from the corresponding properties of \( B \). □

Theorem 4.4. There is a continuous function \( F : \overline{X}_n \times \overline{X}_n \times I \to \overline{X}_n \) such that

1. \( F(T_0, T_1, 0) = T_0 \);
2. \( F(T_0, T_1, 1) = T_1 \);
3. \( F(T_0, T_0, t) = T_0 \) for all \( t \); and
4. \( F \) is equivariant under the diagonal action of Out\( (F_n) \) on \( \overline{X}_n \times \overline{X}_n \).

Proof. Let \( \phi = M(T_0, T_1) \) be the morphism from Proposition 4.3. (Since \( T_1 \) is a minimal action, \( \phi \) is surjective.) There is a natural path in \( \overline{X}_n \) from \( T_0 \) to \( T_2 = \mathcal{H}(M(T_0, T_1)) \), since \( T_2 \) is in the closure of the simplex of \( \overline{X}_n \) containing \( T_0 \). Applying Proposition 4.2 to \( \phi \), we obtain a continuous path of nontrivial semisimple actions from \( T_2 \) to \( T_1 \), given by \( \alpha(t) = \mathcal{H}(\phi_{0t}) \), \( 0 \leq t \leq 1 \). This path will not in general be contained in the set \( \mathcal{H} \) of minimal actions. However, \( [I(\alpha(t))] \) is a path from \( T_2 \) to \( T_1 \) which is contained in \( \mathcal{H} \) (by our identification) and, in fact, is contained in \( \overline{X}_n \) [Sk, Theorem 6.7]. The composition of these two paths gives a natural path in \( \overline{X}_n \) from \( T_0 \) to \( T_1 \). If \( M(T_0, T_0) \) is the identity morphism, \( F \) is the trivial path at \( T_0 \). Since the morphism \( M(T_0, T_1) \) is equivariant under the action of Out\( (F_n) \), so is the path from \( T_0 \) to \( T_1 \), and by Proposition 4.2, it varies continuously with \( T_0 \) and \( T_1 \). □

5. Finite groups of automorphisms

Theorem 5.1. The subset \( X_G \) (resp. \( \overline{X}_G \)) of \( X_n \) (resp. \( \overline{X}_n \)) fixed by a finite subgroup \( G \) of Out\( (F_n) \) is contractible.

Proof. By a theorem of Culler [Cu], \( G \) has a fixed point \( T_0 \) in \( X_n \). By Theorem 4.4, we have a natural path from \( T_0 \) to any other fixed point \( T_1 \). This path is contained in \( X_n \) (resp. \( \overline{X}_n \)) if \( T_1 \) is. By equivariance, each such path is fixed under Out\( (F_n) \). Since these paths vary continuously with \( T_1 \), we can contract \( X_G \) (resp. \( \overline{X}_G \)) continuously along these paths to the point \( T_0 \). □

Remarks. 1. The same argument shows that the fixed point set of any finite subgroup of Out\( (F_n) \) in the space of all nontrivial semisimple actions of \( F_n \) on \( \mathbb{R} \)-trees is contractible.
2. For a finite subgroup $G$ of $\text{Out}(F_n)$, Krstić and Vogtmann [KV] construct a subcomplex $L_G$ of $X_n$ on which the centralizer $C(G)$ acts with finite stabilizers and finite quotient. Their main result is the contractibility of $L_G$ and hence an upper bound on the vcd of $C(G)$. Since $L_G$ is an equivariant deformation retract of the fixed set $X_G$, Theorem 5.1 yields an alternate route to the contractibility of $L_G$.

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