ON THE ENTROPY NORM SPACES AND THE HARDY SPACE $\text{Re} \, H^1$

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Abstract. R. Dabrowski introduced certain natural multiplier operators which map from the entropy norm spaces of B. Korenblum into the Hardy space $\text{Re} \, H^1$. We show that the images of the entropy norm spaces in $\text{Re} \, H^1$ do not include all of that space.

1. Introduction

We consider the entropy norm spaces of Korenblum [4]. He defined an entropy function $\kappa : [0, 1] \to [0, 1]$ to be a concave, continuous, increasing function with $\kappa(0) = 0$. We denote by $K_0$ the set of such functions such that $\kappa'(0) = \lim_{x \to 0^+} \frac{\kappa(x)}{x} = \infty$. According to Dabrowski [1] to each $\kappa \in K_0$ there is a unique probability measure $\mu = \mu_\kappa$ such that

$$\kappa(x) = \int_0^x \int_I \frac{d\mu(u)}{u} \, du.\$$

Then the entropy norm of a continuous 1-periodic function $f \in C(T)$ (where $T = R \mod 1$) is given by

$$\|f\|_\kappa = \int_0^1 \int_T \Omega_I(f) \, dt \frac{d\mu(s)}{s},$$

where $I = [t-s/2, t+s/2]$ and where $\Omega_I(f) = \sup\{|f(u) - f(v)| : u, v \in I\}$. (This norm was introduced by Korenblum [4]; this formula for the norm is due to Dabrowski [4].) We denote by $C_\kappa \subseteq C(T)$ the space of continuous 1-periodic functions of finite entropy norm.

In [2], Dabrowski introduced an operator $T_\kappa : C_\kappa \to \text{Re} \, H^1$, given by

$$T_\kappa f(t) = \int_T \int_0^1 \frac{\chi_I(t)}{s^2} (f(t) - f(I)) \, d\mu(s) \, dx,$$

where $I = [x-s/2, x+s/2], \quad f(I) = \frac{1}{|I|} \int_I f(t) \, dt$ is the average of $f$ over $I$, and $\chi_I$ is the usual characteristic function of $I$. He showed that $T_\kappa$ is a...
multiplier with coefficients

\[ \beta_n = \beta_n(\kappa) = \frac{1}{2\pi^2 n^2} \int_{0,1} (\cos(2\pi ns) - 1 + 2\pi^2 n^2 s^2) \frac{1}{s^3} d\mu_\kappa(s) \]

(for \( n > 0 \) we set \( \beta_{-n} = \beta_n \) and \( \beta_0 = 0 \)). In [3], Dabrowski asked the question: given \( f \in \text{Re} H^1 \), are there \( \kappa \in K_0 \) and \( g \in C_\kappa \) such that \( f = T_\kappa g \)? (One reason why this question is of interest is because, as Dabrowski remarks, a positive answer would imply the Fefferman duality \((\text{Re} H^1(0))^* = \text{BMO})\).

2. The main result

We are ready to give a negative answer to this question.

**Theorem.** There is a function \( f \in \text{Re} H^1 \) such that there are no \( \kappa \in K_0 \) and \( g \in C_\kappa \) with \( f = T_\kappa g \).

**Proof.** We construct \( f \) as follows. Let \( h \) be the function with Fourier series

\[ \sum_{n=1}^{\infty} (\sqrt{n} \log(n + 1))^{-1} e_n \]

where \( e_n = e^{2\pi int} \). Then \( h \in H^2 \). So \( h^2 \in H^1 \) (see, e.g., Zygmund [6, VII (7.22), p. 275]). We let \( f = \text{Re}(h^2) \). So of course \( f \in \text{Re} H^1 \). We have

\[ h^2 \sim \sum_{n=1}^{\infty} \left( \sum_{j=1}^{n-1} b_j b_{n-j} \right) e_n \]

where \( b_j = (\sqrt{j} \log(j + 1))^{-1} \). It is not hard to show that \( f \) has Fourier series \( \sum_{n=1}^{\infty} a_n \cos(2\pi nt) \) where \( a_n \geq \text{const.}(\log(n + 1))^{-2} \) for \( n = 1, 2, 3, \ldots \).

Now we suppose that there is a \( \kappa \in K_0 \) and a \( g \in C_\kappa \) such that \( T_\kappa g = f \). We write \( g \) as \( \sum c_n e_n \). Then since \( T_\kappa g = f \) we have \( c_n = a_n/\beta_n \), \( n \geq 1 \). This enables us to write \( g \) as \( \sum_{n=1}^{\infty} c_n \cos(2\pi nt) \) where \( c_n \geq 0 \) for all \( n > 0 \).

We assume that \( g \in C_\kappa \) which implies that \( g \) is bounded. Consequently (since \( g \) has a cosine series with positive coefficients), we must have \( \sum c_n < \infty \) or \( \sum a_n/\beta_n < \infty \). Therefore

(1) \[ \sum_{n=1}^{\infty} \left( \frac{1}{\log(n + 1)} \right)^2 \frac{1}{\beta_n} < \infty. \]

We must also have

(2) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \beta_n < \infty. \]

[By Lang [5], \( \beta_n \) compares with \( n \kappa(1/n) - n^2 \int_0^{1/n} \kappa(t) \, dt = \overline{\kappa}'(1/n) \) where \( \overline{\kappa}(x) = \frac{1}{x} \int_0^x \kappa(t) \, dt \). We have \( \overline{\kappa}(x) = \int_0^x \overline{\kappa}'(t) \, dt \), so this integral must be convergent; we may estimate this integral by the sum

\[ \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n + 1} \right) \overline{\kappa}' \left( \frac{1}{n} \right) \approx \sum_{n=1}^{\infty} \frac{1}{n^2} \beta_n. \]

(Note that \( \overline{\kappa}'(x) = (1/x^2)(x \kappa(1/x) - \int_0^x \kappa(t) \, dt) \) is the product of \( 1/x^2 \) and a function which goes to 0 monotonically as \( x \to 0 \). So the integral and the sum compare.)]
But (1) and (2) are not compatible. Indeed, suppose the sums (1) and (2) are both finite. Then by the Cauchy-Schwarz inequality
\[
\sum_{n=1}^{\infty} \frac{1}{n} \log(n+1) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sqrt{\beta_n} \right) \left( \frac{1}{\log(n+1)} \frac{1}{\sqrt{\beta_n}} \right) \\
\leq \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \beta_n \right)^{1/2} \left( \sum_{n=1}^{\infty} \left( \frac{1}{\log(n+1)} \right)^2 \frac{1}{\beta_n} \right)^{1/2} < \infty,
\]
which is nonsense. So there cannot be \( \kappa \in K_0, \ g \in C_\kappa \) such that \( T_\kappa g = f \), and we are done. \( \square \)

REFERENCES


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