THE CHARACTERISTIC EXPONENT
OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS
WITH TWO IRREGULAR SINGULAR POINTS

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ABSTRACT. A second-order linear differential equation with two irregular singular points, of which one has unit rank, is investigated. A simple limit formula is obtained for the characteristic exponent of the multiplicative solutions as well as a more detailed asymptotic formula which is suitable for getting accurate numerical values. Only the recursively known coefficients of the formal power series solutions at the irregular singular point of unit rank enter our formulas.

1. INTRODUCTION

The differential equation under consideration is

\[ z^2 f'' + zf' - \left[ z^2 + Bz + L^2 + \sum_{m=1}^{M} D_m z^{-m} \right] f(z) = 0, \]

where for simplicity of presentation the \( M + 2 \) parameters \( B, L, D_1, \ldots, D_M \) are assumed to be real.

Relative to the irregular singular point of unit rank at infinity, there are formal power series solutions of (1.1),

\[ f_{\infty 1}(z) = \exp(z)z^{-1/2+B/2} \sum_{n=0}^{\infty} a_n(1)n!(2z)^{-n}, \]

\[ f_{\infty 2}(z) = \exp(-z)z^{-1/2-B/2} \sum_{n=0}^{\infty} a_n(-1)n!(-2z)^{-n}, \]
where the coefficients are given recursively by

\[ a_n = a_n(\kappa), \quad \kappa \in \{1, -1\}, \]
\[ a_0 = 1, \]
\[ a_n = \frac{1}{n^2} \left\{ \left( -L - \frac{1}{2} - \frac{1}{2} \kappa B + n \right) \left( L - \frac{1}{2} - \frac{1}{2} \kappa B + n \right) a_{n-1} \right. \]
\[ \left. - \sum_{m=1}^{M} \frac{(2\kappa)^m}{(n-m)^m} D_m a_{n-m-1} \right\}, \]
\[ n = 1, 2, \ldots (a_{-M} = a_{-M+1} = \cdots = a_{-1} = 0). \]

Here and below use is made of the Pochhammer symbol

\[(x)_k = x(x + 1)\cdots(x + k - 1) = \Gamma(x + k)/\Gamma(x).\]

The formal solutions (1.2) are asymptotic expansions as \( z \to \infty \) in suitable sectors of the complex plane.

The local solutions relative to the origin, which is an irregular singular point if \( M > 0 \), are not needed for our purpose.

In the ring-shaped region \( 0 < |z| < \infty \) there are Floquet solutions

\[(1.4) \quad f_\mu(z) = z^\mu \sum_{n=-\infty}^{\infty} c^n \mu z^n, \quad \mu \in \{-\nu, \nu\} \text{ (if } 2\nu \text{ is not an integer)}, \]

where the coefficients obey the recurrence relation

\[(1.5) \quad -c^n_{\mu,-z} - Bc^n_{\mu,n} + (-L + \mu + n)(L + \mu + n)c^n = \sum_{m=1}^{M} D_m c^n_{\mu,n+m} = 0. \]

This recurrence relation may be viewed as an infinite homogeneous system of linear equations. The requirement that (after the equations have been divided by appropriate \( n \)-dependent factors to ensure convergence) its determinant should vanish determines the possible values of the characteristic exponent (or circuit exponent or Floquet exponent) \( \mu \). Infinite determinants of this type also occur in the context of Hill’s differential equation and, therefore, are known as Hill determinants [14]. It has been shown that \( \mu \) satisfies a certain transcendental equation which contains the Hill determinant for \( \mu = 0 \) only [14]. To evaluate this Hill determinant numerically as the limit \( N \to \infty \) of cut-off \( N \) by \( N \) determinants has become feasible by the work of Mennicken and Wagenführer [8, 9] who investigated the asymptotic behavior as \( N \to \infty \) and then were able to improve the speed of convergence considerably [12].

Another method for computing the characteristic exponent uses numerical integration of the differential equation [13].

This paper reports an entirely different method for evaluating the characteristic exponent. We obtain an essentially explicit formula in terms of the recursively available coefficients \( a_n(\kappa) \) of the formal solutions (1.2) at the irregular singular point of unit rank.

2. Contour integral solutions

The type of the singular point at infinity suggests a Laplace integral representation of the solutions. Extracting first an arbitrary power of \( z \) for later
flexibility, we consider solutions

\begin{equation}
 f(z) = z^k(2\pi i)^{-1} \int_C \exp(zt) V(t) \, dt.
\end{equation}

Then the weight function \( V(t) \) is found, by standard techniques, to be a solution of the \( t \)-equation

\begin{equation}
 (t^2 - 1)V^{(M+2)} + \{(2M + 3 - 2\lambda)t + B\} V^{(M+1)}
 + (M + 1 - \lambda - L)(M + 1 - \lambda + L)V^{(M)} - \sum_{m=1}^{M} (-1)^m D_m V^{(M-m)} = 0,
\end{equation}

and the possible contours \( C \) are such that the bilinear concomitant [2] (that is the integrated term) has the same value, identically in \( z \), at the start and at the end of the contour. This lengthy expression, the details of which are not needed here, contains a common factor \( \exp(zt) \) which dominates its behavior when \( t \to \infty \). So there are permissible contours which start at and return to infinity in suitable directions of the \( t \)-plane depending on \( \arg(z) \).

3. Floquet solutions of the \( t \)-equation

The \( t \)-equation (2.2) has two regular singular points at \( t = -1 \) and \( t = 1 \) and one irregular singular point at infinity. Outside the unit circle we have Floquet solutions

\begin{equation}
 V_\mu(t) = t^{\mu-1} \sum_{n=-\infty}^{+\infty} d_n^\mu t^{-n},
\end{equation}

where the coefficients obey the recurrence relation

\begin{equation}
 - (\mu - \lambda + n - 1)M+2 d_{n+2}^\mu - B(\mu - \lambda + n)M+1 d_{n+1}^\mu 
 + (\mu - \lambda + n + 1)M(-L + \mu + n)(L + \mu + n) d_n^\mu 
 - \sum_{m=1}^{M} (\mu - \lambda + n + 1 + m)M-m D_m d_{n+m}^\mu = 0,
\end{equation}

which, by comparison with (1.5), is satisfied if

\begin{equation}
 d_n^\mu = \Gamma(\mu - \lambda + 1 + n) c_n^\mu.
\end{equation}

So the integral representation (2.1) with the weight function (3.1) and an infinite contour which surrounds both the finite singular points essentially yields the Floquet solutions (1.4) of the \( z \)-equation (1.1), and \( \mu \) is the characteristic exponent as before.

The recurrence relation is also satisfied if all the coefficients with positive indices \( n > 0 \) vanish and \( \mu = \lambda - 1, \lambda - 2, \ldots, \lambda - M - 2 \) is one of the roots of the corresponding \( (M + 2) \)-th order indicial equation. Consequently, there is another solution

\begin{equation}
 V_{\lambda-1}(t) = \sum_{n=0}^{\infty} d_{\lambda-n}^{\lambda-1} t^n.
\end{equation}

While here the coefficients with \( n = 0, 1, \ldots, M - 1 \) are arbitrary constants of integration, so that (3.4) represents an \( M \)-dimensional solution space, the
coefficients with \( n = M, \ M + 1 \) have to be chosen in such a way that the series (3.4) converges outside the unit circle; but, then it converges also on and inside the circle and represents an entire function. We do not need this solution, which does not contribute to the contour integrals under consideration.

4. Solutions relative to the regular singular points of the \( t \)-equation

The exponents of the \( t \)-equation relative to the singular points \( t = \kappa, \) where \( \kappa = \pm 1, \) are \( 0, 1, \ldots, M, \ \lambda - 1/2 - \kappa B/2. \) Provided that \( \lambda - \kappa B/2 \) is not equal to half an odd integer, the solutions can be written as

\[
V(1, t) = F(1, 1 - t), \quad |t - 1| < 2, \\
U_j(1, t) = G_j(1, 1 - t), \quad |t - 1| < 2, \ j = 0, 1, 2, \ldots, M,
\]

and

\[
V(-1, t) = F(-1, 1 + t), \quad |t + 1| < 2, \\
U_j(-1, t) = G_j(-1, 1 + t), \quad |t + 1| < 2, \ j = 0, 1, 2, \ldots, M,
\]

where

\[
G_j(\kappa, x) = x^j \sum_{n=0}^{\infty} A_n(\kappa, j)x^n, \quad |x| < 2,
\]

\[
F(\kappa, x) = x^{\lambda - 1/2 - \kappa B/2}H(\kappa, x),
\]

\[
H(\kappa, x) = \sum_{n=0}^{\infty} A_n(\kappa, \lambda - 1/2 - \kappa B)x^n, \quad |x| < 2,
\]

with the initial coefficients chosen arbitrarily as

\[
A_0(\kappa, q) = 1 \quad \text{for } q = 0, 1, \ldots, M, \ \lambda - 1/2 - \kappa B,
\]

\[
A_j(\kappa, l) = 0 \quad \text{for } l = 0, 1, \ldots, M - 1, \ j = 1, 2, \ldots, M - l.
\]

The other coefficients then are determined by the recurrence relation

\[
A_n(\kappa, q) = \frac{(q + n - \lambda - L)(q + n - \lambda + L)}{2(q + n)(q + n - \lambda + 1/2 + \frac{1}{2}\kappa B)}A_{n-1}(\kappa, q)
\]

\[
- \sum_{m=1}^{M} \frac{1}{2(q + n - m)(q + n - \lambda + 1/2 + \frac{1}{2}\kappa B)}\kappa^m D_m A_{n-m-1},
\]

where \( n > 0 \) if \( q = M, \ \lambda - 1/2 - \frac{1}{2}\kappa B; \ n > j \) if \( q = M - j, \ j = 1, 2, \ldots, M; \) and \( A_{-1}(\kappa, q) = \cdots = A_{-M}(\kappa, q) = 0. \)

By comparison with (1.3) of (4.10) for \( q = \lambda - 1/2 - \frac{1}{2}\kappa B \) we may find

\[
A_n(\kappa, \lambda - 1/2 - \frac{1}{2}\kappa B) = \frac{\Gamma(\lambda + 1/2 - \frac{1}{2}\kappa B)n!}{\Gamma(\lambda + 1/2 - \frac{1}{2}\kappa B + n)}2^{-n}a_n(\kappa).
\]
So the integral representation (2.1) with the weight functions (4.1) or (4.3), respectively, and an infinite contour which surrounds the corresponding singular point may be used to define the solutions of the $z$-equation (1.1) which (apart from constant factors) are asymptotically represented by the formal solutions (1.2).

5. Analytic continuation in the $\tau$-plane

By equations (4.1), (4.2) and (4.3), (4.4) we have two fundamental sets of solutions, valid in different but overlapping domains. Any solution of one set may be expressed as a linear combination of the solutions of the other set; in particular,

\begin{align}
V(\kappa, t) &= E(-\kappa)V(-\kappa, t) + \sum_{j=0}^{M} B_j(-\kappa)U_j(-\kappa, t) .
\end{align}

The coefficients $E$ and $B_j$ might be determined, in terms of $M + 2$ by $M + 2$ determinants the elements of which are Taylor series at half the convergence radius, by evaluating this equation and its first $M + 1$ derivatives at $\tau = 0$. For our purpose we shall really need $E$ only, and below we shall find another method for calculating it. Introducing

\begin{align}
G(-\kappa, x) &= \sum_{j=0}^{M} B_j(-\kappa)G_j(-\kappa, x) ,
\end{align}

\begin{align}
U(1, t) &= G(1, 1 - t) ,
\end{align}

\begin{align}
U(-1, t) &= G(-1, 1 + t)
\end{align}
we have

\begin{align}
V(\kappa, t) &= E(-\kappa)V(-\kappa, t) + U(-\kappa, t) .
\end{align}

It then follows, for consistency of (5.5) with $\kappa = 1$ or $\kappa = -1$, that

\begin{align}
U(\kappa, t) &= \{1 - E(-\kappa)E(\kappa)\}V(-\kappa, t) - E(\kappa)U(-\kappa, t) .
\end{align}

6. Multiplicative solutions in the $\tau$-plane

Let us consider a path in the $\tau$-plane, in the form of a simple closed loop surrounding the two regular singular points $-1, 1$. We want to construct a multiplicative solution, such that it is reproduced, apart from a constant factor, after the loop has been traversed. For this purpose the loop may be viewed as consisting of small circles around the singular points and straight line segments along the real axis. In a neighborhood of the origin let us define

\begin{align}
W(t) &= \alpha V(1, t) + \gamma U(1, t) .
\end{align}

According to (5.5), (5.6) we then also have

\begin{align}
W(t) &= \{\alpha E(-1) + \gamma[1 - E(-1)E(1)]\}V(-1, t)
\end{align}

\begin{align}
&\quad + \{\alpha - \gamma E(1)\}U(-1, t)
&= : W_S(t) .
\end{align}
When we start here and follow the loop in the positive sense, arg$(1-t)$ increases along the circle around the point $+1$ by $2\pi$, so that we obtain

\begin{equation}
W(t) = \alpha \exp(2\pi i[\lambda - \frac{1}{2} - \frac{1}{2}B])V(1, t) + \gamma U(1, t)
\end{equation}

or, according to (5.5), (5.6),

\begin{equation}
W(t) = \{\alpha E(-1) \exp(2\pi i[\lambda - \frac{1}{2} - \frac{1}{2}B]) + \gamma [1 - E(-1)E(1)]\} V(-1, t) + \{\alpha \exp(2\pi i[\lambda - \frac{1}{2} - \frac{1}{2}B]) - \gamma E(1)\} U(-1, t).
\end{equation}

Following the loop further, arg$(1+t)$ is changed along the circle around the point $-1$ by $2\pi$, so that we obtain

\begin{equation}
W(t) = \{\alpha E(-1) \exp(2\pi i[\lambda - \frac{1}{2} - \frac{1}{2}B])
\begin{align*}
&+ \gamma [1 - E(-1)E(1)]\} \exp(2\pi i[\lambda - \frac{1}{2} + \frac{1}{2}B])V(-1, t) \\
&+ \{\alpha \exp(2\pi i[\lambda - \frac{1}{2} - \frac{1}{2}B]) - \gamma E(1)\}U(-1, t)
\end{align*}
\end{equation}

\begin{equation}
=: W_E(t).
\end{equation}

Now $W(t)$ is a multiplicative solution if near the starting point and near the end of the loop it is the same function apart from a constant factor, say,

\begin{equation}
W_E(t) = p \exp(2\pi i\lambda)W_5(t),
\end{equation}

where the exponential factor is introduced for later convenience. Then $\alpha$ and $\gamma$ satisfy the homogeneous system of linear equations

\begin{equation}
E(-1)\{p - \exp(2\pi i\lambda)\}x + [1 - E(-1)E(1)]\{p + \exp(\pi iB)\}x = 0,
\end{equation}

\begin{equation}
-p + \exp(-\pi iB)\}x + E(1)\{p - \exp(-2\pi i\lambda)\}x = 0,
\end{equation}

where certain common exponential factors have been omitted. The requirement that the determinant be zero leads to

\begin{equation}
p^2 + 2\{\cos(\pi B)[1 - E(-1)E(1)] - \cos(2\pi \lambda)E(-1)E(1)\}p + 1 = 0.
\end{equation}

The roots $p = p_1, p_2$ of this equation satisfy the relations

\begin{equation}
p_1 + p_2 = -2\{\cos(\pi B)[1 - E(-1)E(1)] - \cos(2\pi \lambda)E(-1)E(1)\},
\end{equation}

\begin{equation}
p_1p_2 = 1.
\end{equation}

Because of the last equation, the two roots may conveniently be represented by means of one (not necessarily real) parameter $\nu$ in the form

\begin{equation}
p_1 = \exp(-2\pi i\nu), \quad p_2 = \exp(2\pi i\nu).
\end{equation}

Then we have

\begin{equation}
p_1 + p_2 = 2\cos(2\pi \nu)
\end{equation}

and, by comparison with (6.9),

\begin{equation}
\cos(2\pi \nu) = -\cos(\pi B) + [\cos(2\pi \lambda) + \cos(\pi B)]E(-1)E(1)
\end{equation}

or, by means of an elementary trigonometric formula,

\begin{equation}
\cos(\pi[\nu + \frac{1}{2}B])\cos(\pi[\nu - \frac{1}{2}B])
\end{equation}

\begin{equation}
= E(-1)E(1)\cos(\pi[\lambda + \frac{1}{2}B])\cos(\pi[\lambda - \frac{1}{2}B]).
\end{equation}
It is advantageous to introduce
\begin{equation}
(6.15) \quad e(\kappa) = \frac{\pi}{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2}B)\Gamma(-\lambda + \frac{1}{2} + \frac{1}{2}B)} E(\kappa),
\end{equation}
a quantity which is independent of \( \lambda \), as will be shown below. Then (6.14) becomes
\begin{equation}
(6.16) \quad \cos(\pi[\nu + \frac{1}{2}B]) \cos(\pi[\nu - \frac{1}{2}B]) = e(-1)e(1)
\end{equation}
or
\begin{equation}
(6.17) \quad [\cos(\pi\nu)]^2 = [\sin(\frac{1}{2}\pi B)]^2 + e(-1)e(1).
\end{equation}
The multiplicative solutions under consideration must be proportional to the Floquet solutions (3.1), which when the loop is traversed gain a factor \( \exp(2\pi i[\lambda - \mu]) \). By comparison with (6.6) and (6.11) \( \nu \) and \( -\nu \) are seen to be characteristic exponents and so in (6.17) \( \nu \) may be replaced by \( \mu \).

7. The characteristic exponent

The theory of the contour integral solutions offers new ways for calculating the characteristic exponent. We may use (6.17) as soon as we can evaluate the quantities \( e \) or, by (6.15), \( E \), which can be determined from (5.5) by asymptotic methods. Darboux's method [10] works as follows. The left-hand side of (5.5) is
\begin{equation}
(7.1) \quad (1 - \kappa t)^{\lambda - 1/2 - \kappa B/2} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B)n!}{\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)} 2^{-n}a_n(\kappa)(1 - \kappa t)^n
\end{equation}
by means of (4.11). The leading singular term, when \( t \to -\kappa \), on the right-hand side of (5.5) is
\begin{equation}
(7.2) \quad E(-\kappa)(1 + \kappa t)^{\lambda - 1/2 + \kappa B/2},
\end{equation}
which, when expanded in powers of \( 1 - \kappa t \) by means of the formula
\begin{equation}
(7.3) \quad (1 - z)^{-s} = \sum_{n=0}^{\infty} \frac{\Gamma(s + n)}{\Gamma(s)n!} z^n,
\end{equation}
becomes
\begin{equation}
(7.4) \quad \sum_{n=0}^{\infty} E(-\kappa) \frac{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B)n!} 2^{\lambda - 1/2 + \kappa B/2 - n}(1 - \kappa t)^n.
\end{equation}
The coefficients of this series should agree asymptotically, as \( n \to \infty \), with those of the series (7.1), where \( 1 - \kappa t \) in the power factor in front of the series, when \( t \to -\kappa \), tends to and may be replaced by 2. By comparison of (7.1) and (7.4) we then find, using (6.15),
\begin{equation}
(7.5) \quad e(-\kappa) = \pi \lim_{n \to \infty} \frac{n!n!}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)} 2^{-\kappa B}a_n(\kappa),
\end{equation}
the quantity which enters the formula (6.17) for the characteristic exponent and is independent of \( \lambda \), as already stated above. This statement is now verified by the fact that
\begin{equation}
(7.6) \quad \frac{n!n!}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)} = n^{1 + \kappa B}[1 + O(n^{-1})]
\end{equation}
as \( n \to \infty \).
Limit formulas such as (7.5) have been obtained, apart from factors which tend to 1 as \( n \to \infty \), by Jurkat, Lutz, and Peyerimhoff [3–5], by Hinton [1], and by Kurth and Schmidt [6] on the basis of quite different considerations.

For practical computations, however, the applicability of the limit formula (7.5) is questionable. The \( a_n \) are available recursively from (1.3) but are not known explicitly for large values of \( n \). So the sequence has to be computed member after member, but the limit is approached slowly since the members differ from the limit by terms of the order of \( 1/n \). On the basis of the work of Schäfke and Schmidt [11] it is possible to derive a more detailed asymptotic formula which is useful even from a computational point of view.

Apart from the complication that now \( 1 - \kappa t \) in the power factor on the left-hand side of (5.5), or in front of the sum in (7.1), can no longer be replaced by 2, the procedure is essentially the same as above, but a finite number of singular terms on the right-hand side of (5.5) is taken into account rather than the leading one, (7.2), alone. In this way we may obtain the following result.

**Theorem 1.** The characteristic exponent \( \mu \) of the multiplicative solutions (1.4) of the differential equation (1.1) is given by

\[
(7.7) \quad \cos(\pi \mu)^2 = [\sin(\frac{1}{2}\pi B)]^2 + e(-1)e(1),
\]

where

\[
(7.8) \quad e(-\kappa) = \pi C_n(\kappa) \left\{ 1 + \sum_{k=1}^{K} \frac{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B)_k}{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B - n)_k} h_k(-\kappa) + O(n^{-K-1}) \right\}^{-1}
\]
as \( n \to \infty \), with

\[
(7.9) \quad C_n(\kappa) = \frac{n!n!}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)} 2^{-\kappa B} a_n(\kappa),
\]

\[
(7.10) \quad h_k(-\kappa) = \frac{(\lambda - \frac{1}{2} - \frac{1}{2}\kappa B)_k}{k!}
\]

\[
\times \sum_{l=0}^{k} \frac{(-k)_l!}{(-\lambda + \frac{1}{2} + \frac{1}{2}\kappa B - k)_l(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B)_l} a_l(-\kappa),
\]

and the \( a_n(\kappa) \) are the coefficients, known recursively according to (1.3), in the formal power series solutions (1.2) at the irregular singular point of unit rank.

With a suitable choice of \( n \) and \( K \) (for which there is much freedom) this theorem yields accurate values of the characteristic exponent. As long as \( n \) is finite, the right-hand side of (7.8) depends on \( \lambda \), which is quite arbitrary except that \( \lambda - B/2 \) or \( \lambda + B/2 \) must not be equal to (or, for computational reasons, close to) half an odd integer. Thus, various different values of \( \lambda \) may be used, and comparison of the corresponding results (which asymptotically are independent of \( \lambda \)) may give an idea of the achieved accuracy. From a computational point of view, however, the choice \( \lambda = L \) (or \( \lambda = -L \)) is most advantageous, as will be shown below.

**8. More detailed proof of Theorem 1**

In order to derive the more detailed asymptotic formula (7.8) for \( E(-\kappa) \) or \( e(-\kappa) \), we first have to multiply (5.5) by
(8.1) \[\frac{1}{2}(1 - \kappa t)^{-(\lambda - 1/2 - \kappa B/2)} = \sum_{l=0}^{\infty} \frac{(\lambda - \frac{1}{2} - \frac{1}{2}\kappa B)l}{l!} \left[\frac{1}{2}(1 + \kappa t)\right]^l,\]

using the left-hand side of (8.1) on the left and the right-hand side of (8.1) on the right of (5.5). On the left we then have

(8.2) \[2^{\lambda - 1/2 - \kappa B/2} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B)l}{\Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B + n)} 2^{-n} a_n(\kappa)(1 - \kappa t)^n,
\]

and on the right, including now the next \(K\) singular terms in addition to the leading one, we have

(8.3) \[E(-\kappa) \sum_{k=0}^{K} 2^{-k} h_k(-\kappa)(1 + \kappa t)^{\lambda - 1/2 + \kappa B/2 + k},\]

where

(8.4) \[h_k(-\kappa) = \sum_{l=0}^{k} \frac{(\lambda - \frac{1}{2} - \frac{1}{2}\kappa B)_{k-l} l!}{(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B)_{(k-l)} l!} a_l(-\kappa).
\]

Expanded in powers of \(1 - \kappa t\), (8.3) becomes

(8.5) \[E(-\kappa) \sum_{n=0}^{\infty} \sum_{k=0}^{K} h_k(-\kappa) \frac{(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B - k)_n}{n!} 2^{\lambda - 1/2 + \kappa B/2 - n}(1 - \kappa t)^n,
\]

or

(8.6) \[E(-\kappa) \sum_{n=0}^{\infty} 2^{\lambda - 1/2 + \kappa B/2 - n} \frac{(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B)_n}{n!} \times \sum_{k=0}^{K} \frac{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B)_{k} h_k(-\kappa)(1 - \kappa t)^n}{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B - n)_{k}}
\]

by means of the identity

(8.7) \[\frac{(x - k)_n}{(x)_n} = \frac{(1 - x)_k}{(1 - x - n)_k}.
\]

Asymptotically, as \(n \to \infty\), the coefficients of the power series in \(1 - \kappa t\) on the left and on the right should agree, up to and including terms of order \(n^{-K}\) corresponding to the fact that the next \(K\) singular terms are taken into account. Comparison of (8.2) and (8.6) then yields

\[E(-\kappa) = \Gamma(-\lambda + \frac{1}{2} - \frac{1}{2}\kappa B) \Gamma(\lambda + \frac{1}{2} - \frac{1}{2}\kappa B) C_n(\kappa)
\]

(8.8) \[\times \left\{ \sum_{k=0}^{K} \frac{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B)_{k}}{(\lambda + \frac{1}{2} + \frac{1}{2}\kappa B - n)_{k}} h_k(-\kappa) + O(n^{-K-1}) \right\}^{-1}
\]

with the coefficients \(C_n\) of (7.9). Writing the term with \(k = 0\), which is equal to 1, separately and using (6.15) we get (7.8). Rewriting (8.4) by means of the identity

(8.9) \[\frac{(x)_{k-l}}{(k-l)!} = \frac{(x)_{k}(-k)_{l}}{k!(1-x-k)_{l}}
\]

yields (7.10).
9. Choice of the computational parameter \( \lambda \)

The question remains if there is a particularly advantageous value of \( \lambda \). The leading contribution to the remainder term in (7.8) comes from the first term omitted corresponding to \( k = K + 1 \). In order to evaluate it, at least approximately, we take the first term on the right-hand side of the recurrence relation for the \( a_n \) in (1.3) into account only. We then obtain

\[
a_n(\kappa) \approx \frac{(-L + \frac{1}{2} - \frac{1}{2} \kappa B)_n(L + \frac{1}{2} - \frac{1}{2} \kappa B)_n}{n! n!}.
\]

On the basis of this approximation we get, from (7.10),

\[
h_k(-\kappa) \approx \frac{(\lambda - \frac{1}{2} - \frac{1}{2} \kappa B)_k}{k!} 3 F_2 \left( \begin{array}{c} -k, -L + \frac{1}{2} + \frac{1}{2} \kappa B, L + \frac{1}{2} + \frac{1}{2} \kappa B \\ -\lambda + \frac{1}{2} + \frac{1}{2} \kappa B - k, \lambda + \frac{1}{2} + \frac{1}{2} \kappa B \end{array} \right) \left( 1 \right)
\]
or

\[
h_k(-\kappa) \approx \frac{(\lambda + L)_k(\lambda - L)_k}{k!(\lambda + \frac{1}{2} + \frac{1}{2} \kappa B)_k}
\]

by Saalschütz's formula [7]. So the leading contributions to the coefficients \( h_k \) are seen to vanish for \( \lambda = L \) or for \( \lambda = -L \), and this is the case for all the \( k > 0 \), in particular, for \( k = K + 1 \). To this behavior of the \( h_k \) corresponds the behavior of the \( C_n \). Again on the basis of (9.1), we obtain, from (7.9),

\[
C_n(\kappa) \approx \frac{2^{-\kappa B}}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)\Gamma(\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)} \times \frac{(-L + \frac{1}{2} - \frac{1}{2} \kappa B)_n(L + \frac{1}{2} - \frac{1}{2} \kappa B)_n}{(-\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)_n(\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)_n},
\]

which becomes independent of \( n \) for \( \lambda = L \) or for \( \lambda = -L \). Since (9.1) is exact for \( n = 1 \), (9.3) and (9.4) are also exact for \( k = 1 \) or \( n = 1 \), respectively.

10. An example

It might be instructive to look at the approximate equations of the preceding section a little bit more. They yield

\[
e(-\kappa) = \frac{\pi 2^{-\kappa B}}{\Gamma(-\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)\Gamma(\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)} \times \frac{(-L + \frac{1}{2} - \frac{1}{2} \kappa B)_n(L + \frac{1}{2} - \frac{1}{2} \kappa B)_n}{(-\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)_n(\lambda + \frac{1}{2} - \frac{1}{2} \kappa B)_n} \times \left\{ 1 + \sum_{k=1}^{\infty} \frac{(\lambda + L)_k(\lambda - L)_k}{k!(\lambda + \frac{1}{2} + \frac{1}{2} \kappa B - n)_k} + O(n^{-\kappa - 1}) \right\}^{-1}.
\]

This is the exact result for the simpler differential equation

\[
z^2 f'' + zf' - [z^2 + Bz + L^2]f(z) = 0,
\]

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for which the origin is a regular singular point with indices \(-L, L\). The choice \(\lambda = L\) simplifies (10.1) and yields

\[
e(-\kappa) = \frac{\pi^{2-\kappa B}}{\Gamma(-L + \frac{1}{2} - \frac{1}{2}\kappa B)\Gamma(L + \frac{1}{2} - \frac{1}{2}\kappa B)}
\]

so that

(10.4) \(e(-1)e(1) = \cos(\pi[L - \frac{1}{2} B])\cos(\pi[L + \frac{1}{2} B]) = [\cos(\pi L)]^2 - [\sin(\frac{1}{2}\pi B)]^2\)

and, by (7.7),

(10.5) \([\cos(\pi \mu)]^2 = [\cos(\pi L)]^2\), that is, \(\mu = -L, L \pmod{1}\).

Numerically this (extremely simple) case with \(L = 10.75, B = 1.25\), for instance, would look as shown in Table 1. This table may serve as an illustration of the comments immediately following Theorem 1. Also it demonstrates that accurate values of the characteristic exponent can be computed even if the parameter \(L\) is not so small.

11. A MORE GENERAL DIFFERENTIAL EQUATION

The work of the authors mentioned above shows that the limit formulas like (7.5) are valid for the more general differential equation in which \(M\) is infinite and the series in (1.1) convergent as long as \(|z| > R\) for some finite \(R > 0\). If \(R = 0\), this means that the differential equation still has two singular points, but at the origin there is now an irregular singular point of infinite rank, that is an essential singularity. If \(R > 0\), the differential equation is not well specified for \(|z| \leq R\) and may have more than two singular points.

Our proof is based on the Laplace integral representation of the solutions and, since the order of the differential equation for the weight function (2.2) is \(M + 2\), works for any finite \(M\).

Nevertheless Theorem 1 is valid for the more general differential equation, as suggested by the corresponding behavior of the limit formulas. This may be seen as follows. The difference between the infinite case and the case with a large but finite \(M\) lies in the contributions from the parameters \(D_{M+1}, D_{M+2}, \ldots\).
According to (1.3), they influence the $a_n$ with large $n \geq M + 2$ only. Consequently, if $K < M + 2$, the terms of low order in $n^{-1}$ up to and including $n^{-K}$, which explicitly appear in (7.8), are not at all influenced. So there is no reason to insist that $M$ be finite.

REFERENCES


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